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NEW MEASURES OF DIRECTED AND SYMMETRIC-DIVERGENCE BASED ON m PROBABILITY DISTRIBUTIONS

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Vinocha and Goyal[9] had recently defined a generalised measure of directed divergence based on m probability distributions $P_1, P_2, \cdots, P_m$. The present paper gives a new interpretation of this measure in terms of the distribution P^* which is 'closest' to $P_2, P_3, \cdots, P_{m-1}$. It also gives a more general measure of directed divergence and another general measure of symmetric divergence based on $P_1, P_2, \cdots, P_m$. Both these measures have very meaningful and natural interpretations. Relations between these measures and Taneja's measure of information improvement due to N revisions are also established.

1. VINOCHA AND GOYAL'S MEASURE OF GENERALISED DIRECTED DIVERGENCE

Let $P_1, P_2, \ldots, P_m$ be m probability distributions, where

$$P_r = (p_{1r}, p_{2r}, \cdots, p_{mr});\ r = 1, 2, \ldots, m, \qquad \ldots(1)$$

then Vinocha and Goyal[9] defined a measure of generalised directed divergence based on these m probability distributions by

$$D_1(P_1, P_2, \ldots, P_m) = \sum_{i=1}^{n} p_{i1} \ln \frac{p_{ir}}{p_{im}} - \left(- \sum_{\substack{k=2 \\ k \neq r}}^{m-1} \sum_{i=1}^{n} p_{i1} \ln p_{ik} \right) \qquad \ldots(2)$$

$$= (P_1; P_m, P_r) - \sum_{\substack{k=1 \\ k \neq 2}}^{m-1} I(P_1; P_k) \qquad \ldots(3)$$

where $I(P_1; P_m, P_r)$ is Theil's[8] measure of information improvement, when the true distribution is P_1 and its estimate is revised from P_m to P_r and $I(P_1; P_k)$ is Kerridge's[5] measure of inaccuracy of P_1 relative to P_k.

Measure (3) was called "the information in m distributions space, as the first term on the right hand side of (3) is the improvement information of three distributions 1, r and m and the second term is the inaccuracy of the remaining".

It is true that the measure (3) depends on all the m distributions and it can measure some information connected with these distributions, but unless the nature of this information is clear, its characterisation is open to the following criticism of Aczel[1].

"I wish to urge here caution with regard to generalisations in general, and in particular with regard to those introduced through characterisation. In the best of all possible worlds, there is an information measure, which comes from an applied problem, it has interesting properties (usually attractive reasonable generalisations of properties of Shannon entropy or of similar widely used measures) and these properties characterise it. Less ideal, but still acceptable in my opinion is the following situation. Some natural looking weakening or generalisation of the properties characterising Shannon-type measures are isolated and all measures having these properties are determined. If the properties are indeed intuitive and significant, then there is a good chance that the measures thus obtained may have future applications. What many authors seem to do is to contrive some generalisation of known information measures, derive its often not very interesting or natural and also often not very attractive properties and then characterise, by several of these properties the measure which they have defined in the first place. Not many good or useful results can be expected from this kind of activity".

The essence of a measure is not its characterisation, but it lies in the interesting, useful and meaningful properties it may possess.

Let P_1 be the true distribution and let P_m be its initial estimate. Let P_m be revised to P_r, then there is an information improvement

$$I\,(P_1;\, P_m,\, P_r), \qquad \text{...(4)}$$

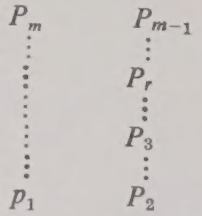

Fig. 1

which gives a measure of how much closer P_r is to P_1 as compared with the 'distance' of P_m from P_1.

However we also know $P_2, P_3, \ldots, P_{r-1}, P_{r+1}, \ldots, P_{m-1}$ and from (4), we subtract the total inaccuracy of P_1 relative to $P_2, P_3, \ldots, P_{r-1}, P_{r+1}, \ldots, P_{m-1}$ to get the measure (3), but it is not clear why we should subtract the inaccuracies and not e.g. the directed divergences.

It also appears from the definition that the measure (3) may depend on the choice of the distribution P_r.

2. An Alternative Interpretation of the Measure

We can simplify (2) to get

$$D_1\,(P_1, P_2, \ldots, P_m) = \sum_{r=2}^{m-1} \sum_{i=1}^{n} p_{i1} \ln p_{ir} - \sum_{i=1}^{n} p_{i1} \ln p_{im}$$

(*equation contiuned on p.* 619)

$$= \sum_{i=1}^{n} p_{i1} \ln \frac{p_{i2}\, p_{i3}, \ldots, p_{im-1}}{p_{im}}$$

$$= (m-2) \sum_{i=1}^{n} p_{i1} \ln (p_{i2}\, p_{i3}, \ldots, p_{im-1})^{\frac{1}{m-2}}$$

$$- \sum_{i=1}^{n} p_{i1} \ln p_{im} \qquad \ldots(5)$$

$$= (m-2) \sum_{i=1}^{n} p_{i1} \ln \frac{p_i^*}{p_{i_m}{}^{1/(m-2)}} \qquad \ldots(6)$$

where p_i^* is the geometric mean of the ith components of distributions $P_2, P_3, \ldots, P_{m-1}$. Let

$$\sum_{i=1}^{n} p_i^* = A \qquad \ldots(7)$$

so that

$$P^* = \left(\frac{p_1^*}{A}, \frac{p_2^*}{A}, \ldots, \frac{p_n^*}{A} \right) \qquad \ldots(8)$$

is a probability distribution. We can write (5) as

$$D_1 (P_1, P_2, \ldots, P_m) = (m-2) \ln A - (m-2)\, I(P_1, P^*) + I(P_1, P_m) \qquad \ldots(9)$$

so that $D_1 (P_1, P_2, \ldots, P_m)$ can be interpreted in terms of the inaccuracies of P_1 relative to P^* and P_m.

The factor $(m-2)$ on the R.H.S. of the second term may be explained on the basis that P^* is based on $(m-2)$ distributions $P_2, P_3, \ldots, P_{m-1}$. We now seek an interpretation for the distribution P^*.

3. An Interpretation of the Distribution P^*

The sum of Kullback-Leibler[6] directed divergences of a distribution P form $P_2, P_3, \ldots, P_m$ is given by

$$\sum_{r=2}^{m-1} D(P : P_r) = \sum_{r=2}^{m-1} \sum_{i=1}^{n} p_i \ln \frac{p_i}{p_{ir}}$$

(*equation continued on p.* 620)

$$= (m-2) \sum_{i=1}^{n} p_i \ln p_i - \sum_{i=1}^{n} p_i \ln p_{i2}, p_{i3}, \ldots, p_{im-1}$$

$$= (m-2) \sum_{i=1}^{n} p_i \ln p_i - (m-2) \sum_{i=1}^{n} p_i \ln p_i^*$$

$$= (m-2) \left[\sum_{i=1}^{n} p_i \left[\ln \frac{p_i}{p_i^* / A} - \ln A \right] \right] \quad \ldots(10)$$

$$= (m-2) \left[\sum_{i=1}^{n} p_i \ln \frac{p_i}{p_i^* / A} - \ln A \right]. \quad \ldots(11)$$

This is minimum when $P = P^*$.

Thus P^* is the distribution, the sum of the directed divergences from which to the $(m-2)$ distributions $P_2, \ldots, P_{m-1}$ is minimum. It is in some sense, the distribution which is closest to the $(m-2)$ distributions.

4. An Alternative Interpretation of $D_1 (P_1, P_2, \ldots, P_m)$

As such from (9), the measure

$D_1 (P_1\ P_2, \ldots, P_m)$ can be interpreted = constant (depending on $P_2, \ldots, P_{m-1}$) $-(m-2)$ inaccuracy of P_1 with respect to P^* + inaccuracy of P_1 with respect to P_m. ...(12)

Now

$$I (P_1, P^*) = H (P_1) + D (P_1, P^*) \quad \ldots(13)$$

$$I (P_1, P_m) = H (P_1) + D (P_1, P_m) \quad \ldots(14)$$

so that (9) gives

$$D_1 (P_1, P_2, \ldots, P_m) = (m-2) \ln A - (m-3) H (P_1) - (m-2) D (P_1\ P^*) + D (P_1, P_m) \quad \ldots(15)$$

which gives an interpretation of $D_1 (P_1, \ldots, P_m)$, but it is not obvious why we should consider the measure at all as a measure of net information in the m distributions.

5. An Alternative Measure of Generalised Directed Divergence

Let P_1 be the true distribution and let its initial estimate be P_m. Let $P_2, P_3, \ldots, P_{m-1}$ be any $m-2$ revised estimates of P_1, then the $(m-2)$ improvements of information are

$$I (P_1; P_m, P_2),\ I (P_1; P_m, P_3), \ldots,\ I (P_1; P_m, P_{m-1})$$

and the average of these is given by

$$D_2(P_1, P_2, \ldots, P_m) = \frac{1}{(m-2)} \sum_{r=2}^{m-1} \sum_{i=1}^{n} p_{i1} \ln \frac{p_{ir}}{p_{im}}$$

$$= \frac{1}{(m-2)} \sum_{i=1}^{n} p_{i_1} \ln \frac{p_{i2}, \ldots, P_{im-1}}{p_{im}^{m-2}}$$

$$= \frac{1}{m-2} \sum_{i=1}^{n} p_{i1} \ln \frac{p_i^{*m-2}}{p_{im}^{m-2}} = \sum_{i=1}^{n} p_{i1} \ln \frac{p_i^*}{p_{im}} \quad \ldots(16)$$

$$= \sum_{i=1}^{n} p_{i1} \ln \frac{p_i^*/A}{p_{im}} + \ln A = I(P_1; P_m, P^*) + \ln A. \quad \ldots(17)$$

It is thus observed that

(i) $D_2(P_1, P_2, \ldots, P_m)$ differs from $D_1(P_2, \ldots, P_m)$ in only having p_i^* in place of $(p_i^*)^{(m-2)}$.

(ii) It has meaningful interpretation in terms of the improvement of information in revising the estimate of P_1 from P_m to P^*, where P^* is the distribution closest to $P_2, \ldots, P_{m-1}$.

(iii) We can define another measure of generalised directed divergence as

$$D_3(P_1, P_2, \ldots, P_m) = I(P_1; P_m, P^*). \quad \ldots(18)$$

(iv) We can define still another measure of generalised directed divergence by

$$D_4(P_1, P_2, \ldots, P_m) = I(P_1; P_m, \bar{P}) \quad \ldots(19)$$

when

$$\bar{P} = \frac{P_2 + P_3 + \quad + P_{m-1}}{m-2}. \quad \ldots(20)$$

(v) In all these measures, $P_2, P_3, \ldots, P_{m-1}$ can be permuted among themselves.

6. Measures of Generalised Symmetric Divergence

In the above measures, we can interchange $P_2, P_3, \ldots, P_{m-1}$ among themselves, but we cannot interchange P_1, P_m between themselves. However consider the measure

$$J_2 (P_1, P_2, \ldots, P_m) = D_2 (P_1, P_2, \ldots, P_{m-1}, P_m) + D_2 (P_m, P_2, \ldots, P_{m-1}, P_1)$$

$$= \sum_{i=1}^{n} p_{i1} \ln \frac{p_i^*}{p_{im}} + \sum_{i=1}^{n} p_{im} \ln \frac{p_i^*}{p_{i1}} \qquad \ldots(21)$$

$$J_3 (P_1, P_2, \ldots, P_m) = D_3 (P_1, \ldots, P_m) + D_3 (P_m, P_2, \ldots, P_{m-1}, P_1)$$

$$= I (P_1; P_m, \bar{P}) + I (P_m; P_1, \bar{P}) \qquad \ldots(22)$$

$$J_4 (P_1, P_2, \ldots, P_m) = D_4 (P_1, \ldots, P_m) + D_4 (P_m, P_2, \ldots, P_{m-1}, P_1)$$

$$= I (P; P_m, \bar{P}) + I (P_m; P_1, \bar{P}). \qquad \ldots(23)$$

These are all symmetric in the sense that P_1, P_m can be interchanged between themselves.

7. Comparison with Taneja's Measure

Taneja's had earlier defined the measure $D_2 (P_1, P_2, \ldots, P_m)$ in the some way as we have done, but he did not express it in terms of the information improvement $I (P_1; P_m, P^*)$ with the result that he characterised it and studied its properties. The exercise is not needed, once it is noted that it is just a measure of information improvement involving three distributions.

Now since the geometric mean of m positive numbers $\leqslant$ their arithmetic mean, we get

$$p_i^* = (p_{i2}\, p_{i3}, \ldots, p_{im-1})^{\frac{1}{m-2}} \leqq (p_{i2} + p_{i3} + \ldots + p_{im-1})/(m-2) \qquad \ldots(24)$$

so that

$$A = \sum_{i=1}^{m} p_i^* \leqslant \frac{1}{m-1} \left(\sum_{i=1}^{m} p_{i2} + \sum_{i=1}^{m} p_{i3} + \ldots + \sum_{i=1}^{m} p_{im-1} \right) = 1 \qquad \ldots(25)$$

and

$$\ln A \leqslant 0. \qquad \ldots(26)$$

Now from (17) and (26)

$$D_2 (P_1, P_2, \ldots, P_m) \leqslant I (P_1; P_m, P^*) = D_3 (P_1, P_2, \ldots, P_m). \qquad \ldots(27)$$

Thus our second measure of generalised directed divergence is always $\leqslant$ our third measure of generalised directed divergence and the two measures will coincide only if

$$p_{i2} = p_{i3} = \ldots = p_{im-1} \text{ for each } i$$

i.e. if

$$P_2 = P_3 = \ldots = P_{m-1}. \qquad \ldots(28)$$

Again

$$I(P_1; P_m, P^*) = D(P_1 : P_m) - D(P_1 : P^*) \qquad \ldots(29)$$

$$I(P_1; P_m, \bar{P}) = D(P_1 : P_m) - D(P_1 : \bar{P}) \qquad \ldots(30)$$

so that

$$I(P_1; P_m, P^*) - I(P_1, P_m, \bar{P}) = D(P_1, \bar{P}) - D(P_1 : P^*). \qquad \ldots(31)$$

Also the average of the directed divergences from $P_2, P_3, \ldots, P_{m-1}$

$$= \frac{1}{m-2}(D(P_1 : P_2) + D(P_1 : P_3) \ldots D(P_1 : P_{m-1}))$$

$$= \sum_{i=1}^{n} p_{i1} \ln \frac{p_{i1}}{p^*} = D(P_1 : P^*) - \ln A \qquad \ldots(32)$$

so that

$$D_3(P_1, P_2, \ldots, P_m) - D_4(P_1, P_2, \ldots, P^*)$$

$$= D(P_1, \bar{P}) - \frac{1}{m-2}(D(P_1 : P_2) + D(P_1 : P_3) + \ldots + D(P_1 : P_{m-1})) - \ln A. \qquad \ldots(33)$$

Thus the difference between D_3 and D_4

$= -\ln A +$ (Directed divergence of P_1 from the mean of $P_2, P_3, \ldots, P_{m-1}$)

$-$ (mean of Directed divergence of P_1 from $P_2, P_3, \ldots, P_{m-1}$) ...(34)

8. General Discussion

To find a measure of information based on m distributions $P_1, P_2, \ldots, P_m$, we replace the distributions $P_2, P_3, \ldots, P_{m-1}$ by a distribution P^* or $\bar{P}$ or some other distribution central to these $m-2$ distributions, so that we get a total of three distributions and we can find a measure of information improvement based on these three distribution to get the measures

$$D_k(P_1, P_2, \ldots, P_{m-1} : P_m) \text{ or } J_k(P_1; P_2, P_3, \ldots, P_{m-1}; P_m),\ k = 1, 2, 3, 4 \qquad \ldots(35)$$

We could have as well reduced these to measures based on two distributions e.g. we can find the directed divergence from P_1 to the arithmetic or geometric or some other mean of $P_2, P_3, \ldots, P_{m-1}$.

We could have also consider the directed divergence between mean distributions of two groups of distributions, say of groups $P_1, P_1, P_2, \ldots, P_r$ and $P_{r+1}, \ldots, P_m$ or we

could not have considered the information improvement between mean distributions of three groups say $P_1, P_2, \ldots, P_r, P_{r+1}, \ldots, P_s$ and $P_{s+1}, \ldots, P_m$ or we could have found the total information improvement from two subgroups $P_1, P_2, \ldots, P_r$ and $P_{r+1}, \ldots, P_s$ to two other groups $P_{s+1}, \ldots, P_t$ and $P_{t+1}, \ldots, P_m$.

However none of the measures is symmetric in all the distribution $P_1, P_2, \ldots, P_m$ since each one of these changes when $P_1, P_2, \ldots, P_m$ are permuted among themselves.

Kapur[3] has given symmetric measures viz.

$$D(P_1 : P^*) + D(P_2 : P^*) + \ldots + D(P_m : P^*) \qquad \ldots(36)$$

or

$$D(P_1 : \bar{P}) + D(P_2 : \bar{P}) + \ldots + D(P_m : \bar{P}). \qquad \ldots(37)$$

Both the measures are $\geqslant 0$ and vanish iff

$$P_1 = P_2 = \ldots = P_m. \qquad \ldots(38)$$

These can serve as measures of discrepancy or dispersion among the m probability distributions.

Other measures proposed by Kapur[3 4] which are based on all the m probability distributions and which cannot be reduced to measures based on two or three probability distribution are

$$1 - \sum_{i=1}^{m} p_{i1}^{a_1} p_{i2}^{a_2} \ldots p_{im}^{a_m} \qquad \ldots(39)$$

or

$$-\ln \sum_{i=1}^{m} p_{i1}^{a_1} p_{i2}^{a_2} \ldots p_{im}^{a_m} \qquad \ldots(40)$$

where each $a_j \geqslant 0$ and $\sum_{j=1}^{m} a_j = 1$.

9. Some Applications of Measures of Directed and Symmetric Divergence

Before concluding, we give some typical application of the measures given in the present paper.

(a) Measure of Difference of opinion in a Group of Persons

Let a finance company have m members in its board of directors. The company has to decide on the proportions of its funds to be invested in n different stocks. Let the ith member ($i = 1, 2, \ldots, m$) propose an investment portfolio

$$P_i = (p_{i1}, p_{i2}, \ldots, p_{in}); \; \sum_{j=1}^{n} p_{ij} = 1, \; p_{ij} \geqslant 0 \qquad \ldots(41)$$

so that p_{ij} denotes the proportion of the company's funds which the ith director propose for investment on the jth stock. In general $P_1, P_2, \ldots, P_m$ will be different since the perceptions of the different directors differ about the profitability of the different stocks. We are required to find a measure of this difference of perception among the directors.

Our measure of symmetric divergence gives a measure of this difference of opinion among the directors. This would vanish iff all the directors agree completely and will be large if they differ widely in this perception.

Similarly our measure will give a good measure of the difference in a cabinet on how to spend a rupee on different items in the budget.

(*b*) *Finding the Consensus in a Group*

To find the consensus in the above example, we find a distribution which is as close to $P_1, P_2, \ldots, P_m$ as possible. This distribution is given by

$$P^* = (p_1^*, p_2^*, \ldots, p_m^*) \qquad \ldots(42)$$

where

$$p_j^* = \frac{(p_{1j} \cdot p_{2j} \cdot p_{3j} \cdot \;\; , p_{mj})^{1/m}}{\sum_{j=1}^{n} (p_{1j} \cdot p_{2j} \cdot p_{3j} \cdot \ldots \; p_{mj})^{1/m}}, \; j = 1, 2, \ldots, n. \qquad \ldots(43)$$

(*c*) *Finding the Consensus Ranking on a Selection Committee*

Let the ith member of a selection committee give marks

$$M_{i1}, M_{i2}, \ldots, M_{in} \qquad \ldots(44)$$

to the n candidates and let

$$p_{ij} = M_{ij} / \sum_{j=1}^{n} M_{ij} \qquad \ldots(45)$$

then P^* gives the consensus marks vector and from it, we can deduce the consensus ranking vector obtained by just adding the marks, in the following respects.

(i) the marks here have been 'normalised' : This is necessary since different members may have different standards of marking.

(ii) instead of using the arithmetic mean of normalised marks, it is suggested here that geometric mean should be used. This is even otherwise indicated since geometric mean is much less affected by extreme values than the arithmetic mean.

(iii) even the geometric means have been normalised, this of course does not affect the consensus ranking.

(*d*) *Generalising the Principle of Minimum Discrimination Information*

According to the principle of minimum discrimination information (MDIP) we have to choose that probability disiribution, out of all those that satisfy given constraints, which is closest to given a priori probability distributions. However we may be given a number of a priori probability distributions $Q_1, Q_2, \ldots, Q_m$. Thus we may obtain estimates of this distribution from m experienced persons, we then get the Generalised Minimum Discrimination Information Principle (GMDIP), according to which we choose that distribution, out of all those distributions which satisfy given constraints, which is closest to $Q_1, Q_2, \ldots, Q_m$. In this case we need a measure of directed divergence from P to $Q_1, Q_2, \ldots, Q_m$ and we have precisely developed such a measure in this paper.

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FIXED POINT THEOREMS FOR SOME NON-SELF MAPPINGS

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Let K be a closed, convex subset of a Hilbert space X. We establish a fixed point theorem for a certain type of non-self map T K into X. These results extend a theorem of Massa and Roux[3] to non-self maps, and generalize a result of Samanta.[4]

Let X denote a complex Hilbert space, K a closed, convex subset of X. A map $T : K \to X$ will be called a G-type non-self map over K if T satisfies

$$\|Tx - Ty\| \leqslant a(x, y)\|x - y\| + b(x, y)\|x - Tx\| + b(y, x)\|y - Ty\| + c(x, y)\|x - Ty\| + c(y,x)\|y - Tx\| \quad \ldots(1)$$

for all $x, y \in K$, where a, b, c are non-negative functions from $X \times X \to \mathbb{R}^+$ satisfying

$$(a + b + c)(x, y) + (b + c)(y, x) \leqslant 1 \quad \ldots(2)$$

and

$$c(x, y) = c(y, x) \text{ for all } x, y \in K. \quad \ldots(3)$$

Our main result is the following.

Theorem—Let X denote a complex Hilbert space, K a bounded closed convex subset of X, $T : K \to X$ a G-type non-self map satisfying (1)—(3),

$$\inf_{x,y \in K} a(x, y) > 0, \quad \ldots(4)$$

and

$$T : \partial K \to K \quad \ldots(5)$$

where ∂K denotes the boundary of K. If either $c \equiv 0$ on K or $\inf\limits_{x,y\in K} c(x, y) > 0$, then T has a fixed point.

The Theorem is proved by making use of the following Lemmas.

Lemma 1—If T satisfies (1)—(5), then

$$\lim \|Tx_n - x_{n+1}\| = 0,$$

where $x_n = (RT)^n x_0$, and $R : X \to K$ is defined by $R(x) = y$, where

$$\|x - y\| = d(x, K),\ x \in X.$$

PROOF : It has been shown in[1] that R is nonexpansive.

If $Tx_{n-1} \in K$, then $x_n = (RT)\,x_{n-1} = Tx_{n-1}$. Therefore, using (1),

$$\begin{aligned}\|x_n - Tx_n\| &= \|Tx_{n-1} - Tx_n\| \\ &\leqslant a\,\|x_{n-1} - x_n\| + b\|\,x_{n-1} - Tx_{n-1}\| + b'\|\,x_n - Tx_n\| \\ &\quad + c\,\|x_{n-1} - Tx_n\| + c\|\,x_n - Tx_{n-1}\|\end{aligned}$$

where a, b, c are evaluated at (x_{n-1}, x_n) and $b'(x,y) = b(y, x)$.
Thus

$$(1 - b' - c)\,\|x_n - Tx_n\| \leqslant (a + b + c)\,\|x_{n-1} - Tx_{n-1}\|$$

which implies by (4), that $\{\|x_n - Tx_n\|\}$ is monotone decreasing, hence convergent.

If $Tx_{n-1} \notin K$ then, from (5), $x_n = (RT)\,x_{n-1}$. Thus, since R is nonexpansive and $Tx_n \in K$,

$$\|x_n - Tx_n\| = \|RTx_{n-1} - RTx_n\| \leqslant \|Tx_{n-1} - Tx_n\|.$$

The convexity of K implies that $x_n = RTx_{n-1}$ is the point on ∂K satisfying

$$\|x_{n-1} - Tx_{n-1}\| = \|x_{n-1} - x_n\| + \|x_n - Tx_{n-1}\|.$$

Therefore

$$\begin{aligned}&\|x_{n-1} - Tx_n\| + \|x_n - Tx_{n-1}\| \\ &\qquad \leqslant \|x_{n-1} - x_n\| + \|x_n - Tx_n\| + \|x_n - Tx_{n-1}\| \\ &\qquad = \|x_{n-1} - Tx_{n-1}\| + \|x_n - Tx_n\|.\end{aligned}$$

Also $\|x_{n-1} - x_n\| \leqslant \|x_{n-1} - Tx_{n-1}\|$, and it follows that $\{\|x_n - Tx_n\|\}$ is non-increasing.
Set

$$p_n = \|x_n - Tx_n\|,\ p = \lim p_n,\ \epsilon_n = \|x_n - x_{n+1}\|,\ \gamma_n = \|Tx_n - x_{n+1}\|,$$

and

$$\eta_n = p_n - p_{n+1}.$$

Then

$$\eta_n = p_n - p_{n+1} = \|x_n - Tx_n\| - \|x_{n+1} - Tx_{n+1}\|.$$

In all cases,

$$\begin{aligned}\|x_{n+1} - Tx_{n+1}\| &\leqslant \|Tx_n - Tx_{n+1}\| \\ &\leqslant a\,\|x_n - x_{n+1}\| + b\,\|x_n - Tx_n\| + b'\,\|x_{n+1} - Tx_{n+1}\| \\ &\quad + c\,\|x_n - Tx_{n+1}\| + c\,\|x_{n+1} - Tx_n\|,\end{aligned}$$

where a, b, b', c are evaluated at (x_n, x_{n+1}).

If $Tx_n \in K$, then

$$(1 - b' - c)\, \|x_{n+1} - Tx_{n+1}\| \leqslant (a + c)\, \|x_n - x_{n+1}\| + b\, \|x_n - Tx_n\|,$$

and

$$\begin{aligned}\eta_n &\geqslant \|x_n - Tx_n\| - [(a + c)\, \|x_n - x_{n+1}\| \\ &\quad + b\, \|x_n - Tx_n\|]/\,(1 - b' - c) \\ &\geqslant [(1 - b' - c - b)\, \|x_n - Tx_n\| \\ &\quad - (a + c)\, \|x_n - x_{n+1}\|/(1 - b' - c).\end{aligned} \qquad \ldots(6)$$

Since

$$\|Tx_n - x_{n+1}\| = \inf_{y \in K} \|Tx_n - y\|, \text{ for } t \in [0, 1],$$

$\|Tx_n - (tx_n + (1 - t)\, x_{n+1})\|^2 \geqslant \|Tx_n - x_{n+1}\|^2$. Since X is a Hilbert space.

$$\begin{aligned}\|Tx_n - (tx_n + (1 - t)\, x_{n+1})\|^2 &= \|Tx_n - x_{n+1} - t\,(x_n - x_{n+1})\|^2 \\ &= \|Tx_n - x_{n+1}\|^2 + t^2\|\, x_n - x_{n+1}\|^2 \\ &\quad + 2t \operatorname{Re} < Tx_n - x_{n+1}, x_{n+1} - x_n > \\ &\geqslant \|Tx_n - x_{n+1}\|^2.\end{aligned}$$

i. e.

$$\operatorname{Re} < Tx_n - x_{n+1}, x_{n+1} - x_n > \geqslant -t\, \|x_{n+1} - x_n\|^2/2.$$

Since this inequality is true for all $0 \leqslant t < 1$,

$$\operatorname{Re} < Tx_n - x_{n+1}, x_{n+1} - x_n > \geqslant 0.$$

Therefore

$$\begin{aligned}\|Tx_n - x_n\|^2 &= \|Tx_n - x_{n+1}\|^2 + \|x_{n+1} - x_n\|^2 \\ &\quad + 2 \operatorname{Re} < Tx_n - x_{n+1}, x_{n+1} - x_n > \\ &\geqslant \|Tx_n - x_{n+1}\|^2 + \|x_{n+1} - x_n\|^2.\end{aligned}$$

$$p_n^2 \geqslant \gamma_n^2 + \epsilon_n^2. \qquad \ldots(7)$$

From (6) and (7)

$$\begin{aligned}\eta_n &\geqslant [(1 - b' - c - b)\, p_n - (a + c)\, \epsilon_n]/\,(1 - b' - c) \\ &\geqslant \left[(1 - b' - c - b)\ p_n - (a + c) \sqrt{p_n^2 - \gamma_n^2}\,\right]/\,(1 - b' - c)\end{aligned}$$

(equation continued on p. 630)

$$\geqslant (a + c) \left[p_n - \sqrt{p_n^2 - \gamma_n^2}\right] / (1 - b' - c)$$

$$> 0.$$

On the other hand, if $Tx_n \notin K$, then

$$(1 - b' - c) \|x_{n+1} - Tx_{n+1}\| \leqslant a \| x_n - x_{n+1}\| + (b + c) \|x_n - Tx_n\|,$$

and (6) becomes

$$\eta_n \geqslant [(1 - b' - b - 2c) \|x_n - Tx_n\| - a \| x_n - x_{n+1}\|]/(1 - b' - c).$$

Again using (7).

$$\eta_n \geqslant [(1 - b' - b - 2c) b_n - a\epsilon_n] / (1 - b' - c)$$

$$\geqslant a \left[p_n - \sqrt{p_n^2 - \gamma_n^2}\right] / (1 - b' - c).$$

Let $\xi = \inf_{x,y \in K} a/(1 - b' - c)$. From (4), $\xi > 0$, and we have

$$\eta_n \geqslant \xi [p_n^2 - \gamma_n^2] > 0.$$

Since $\{p_n\}$ is convergent, $\lim \eta_n = \lim (p_n - p_{n+1}) = 0$. Therefore $\lim \sqrt{p_n^2 - \gamma_n^2}$ exists. Moreover,

$$\lim \sqrt{p_n^2 - \gamma_n^2} = \lim p_n = p.$$

Therefore $\lim (p_n^2 - \gamma_n^2) = p^2$, which implies $\lim \gamma_n^2 = 0$. Hence $\lim \gamma_n = 0$.

Lemma 2[2]—Let K be a uniformly convex metric space, B_R the closed spherical ball centered at the origin with radius $R > 0$. Let $\epsilon, d > 0$. If $x_1, x_2, x_3 \in B_R$ and satisfy $(d + \epsilon) \geqslant \|x_1 - x_2\| \geqslant d' > 0$ and $(d + \epsilon) \geqslant \|x_2 - x_3\| \geqslant d' > 0$, and if $\|x_2\| \geqslant (1 - \frac{1}{2} \delta (d / R)) R$, then

$$\|x_1 - x_2\| \leqslant \eta \left[1 - \tfrac{1}{2}\delta \left[\frac{d'}{R}\right]\right] (d - \epsilon),$$

there $\eta = \delta^{-1}$ and δ is the modulus of convexity of X.

Lemma 3—If T satisfies the conditions of the Theorem, then

$$\inf_{x \in K} \|x - Tx\| = 0.$$

Proof : Let $d = \inf_{x \in K} \|x - Tx\|$, and assume that $d > 0$. Choose $\epsilon' > 0$ and small enough so that $d' = d - \epsilon' > 0$. Take $\epsilon > 0$ and arbitrary. Then there exists

an $x_0 \in K$ such that $d' \leqslant \|x_0 - Tx_0\| \leqslant d + \epsilon$. Set $x_n = (RT)^n x_0$, $M' = \overline{\lim} \; \|x_n\|$. Since $d > 0$, $M' > 0$. Pick $M'' > M'$. Then $M'/(1 - \delta(d'/M'')/2) > M'$. Choose M to satisfy $M'' > M > M'$ and $M'/(1 - \delta(d'/M'')/2) > M > M'$. Since δ is an increasing function

$$\left[1 - \frac{1}{2}\delta\left[\frac{d'}{M}\right]\right]M < \left[1 - \tfrac{1}{2}\delta\left[\frac{d'}{M''}\right]\right]M < M'.$$

Choose an index n such that $\|x_{n+1}\| \leqslant M, \|x_n\| \leqslant M$, $\|x_{n+1}\| \leqslant M$, $\gamma_{n-1} < \epsilon'$ and

$$\|x_n\| > \left[1 - \tfrac{1}{2}\delta\left[\frac{d'}{M}\right]\right]M.$$

Case I—$Tx_{n-1} \neq RTx_{n-1} = x_n$. Then $\|x_{n+1} - x_n\| = \|Tx_n - x_n\| \leqslant \|Tx_{n-1} - x_{n-1}\| \leqslant \|x_n - x_{n-1}\| + \|x_n - Tx\|_{-1}\|$. This in turn implies that $\|x_{n+1} - x_n\| \geqslant d \geqslant d'$ and $\|x_n - x_{n-1}\| \geqslant d'$. Since $\{\|x_n - Tx_n\|\}$ is decreasing, $\|x_n - x_{n+1}\| = \| Rx_n - RTx_n \| \leqslant \|x_n - Tx_n\|$. Therefore $d' \leqslant | x_{n+1} - x_n\| \leqslant d + \epsilon$ and $d' \leqslant \|x_n - x_{n-1}\| \leqslant d + \epsilon$. Ayplying Lemma 2,

$$\|x_{n+1} - x_{n-1}\| \leqslant \eta\left[1 - \tfrac{1}{2}\delta\left[\frac{d'}{M}\right]\right](d + \epsilon). \qquad \ldots(8)$$

Case II—Case I does not occur. A similar argument yields (8).

Suppose $c \equiv 0$ on K. Define $m = (x_n + x_{n+1})/2$

$$\begin{aligned}\|m - T_m\| &\leqslant (\|x_n - Tm\| + \|x_{n+1} - Tm\|)\\ &\leqslant (\|Tx_{n-1} - Tm\| + \|Tx_n - Tm\| + \gamma_{n-1} + \gamma_n)/2\\ &\leqslant (\|Tx_{n-1} - Tm\| + Tx_n - Tx_{n-1}\| + b'\,\|m - Tm\|\\ &\qquad + \mathbf{a}\,\| xn - m\| + \mathbf{b}\,\|x_n - Tx_n\|\\ &\qquad + \mathbf{b}'\,\|m - Tm\| + \gamma_{n-1} + \gamma_n)/2,\end{aligned}$$

where a, b, b', c are evaluated at (x_{n-1}, m) and $\mathbf{a}, \mathbf{b}, \mathbf{b}', \mathbf{c}$ are the evaluations at (x_n, m). $\|x_{n-1} - m\| \leqslant (\|x_{n-1} - x_n\| + \|x_{n+1} - x_{n+1}\|)/2$ and $\|xn - m\| = \|x_n - x_{n+1}\|/2$. Substituting the above inequality yields

$$\begin{aligned}(2 - b' - \mathbf{b}')\,\|m - Tm\| &\leqslant (a/2 + b + \mathbf{a}/2 + \mathbf{b})\,(d + \epsilon)\\ &\qquad + a\,\|x_{n-1} - x_{n+1}\|/2 + \gamma_{n-1} + \gamma_n.\end{aligned}$$

Use (8) with

$$\zeta = \tfrac{1}{2}\eta\left[1 - \tfrac{1}{2}\delta\left[\frac{d'}{A}\right]\right] < 1$$

where A is chosen large enough so that ζ does not depend on ϵ. Then, since $d \leqslant \|m - Tm\|$,

$$(2 - b - b' - 1/2 - \mathbf{b} - \mathbf{b}' - \mathbf{a}/2)d$$
$$\leqslant a\,\zeta d + (a/2 + b + \mathbf{a}/2 + \mathbf{b} + a\zeta)\,\epsilon + \gamma_{n-1} + \gamma_n$$

which implies that

$$(a + \mathbf{a})\,d/2 \leqslant a\,\zeta d + (a/2 + b + \mathbf{a}/2 + \mathbf{b} + a\zeta)\,\epsilon + \gamma_{n-1} + \gamma_n.$$

Taking the $\sup\limits_{x,y\in K}$ of both sides yields

$$\sup_{x,y\in K} a\,(xy)\,d \leqslant \sup_{x,y\in K} a\,(xy)\,\zeta d$$
$$+ \sup_{x,y\in K} (a/2 + b + \mathbf{a}/2 + b + a\mathbf{s})\,(x, y)\,\epsilon$$
$$+ \gamma_{n-1} + \gamma_n.$$

Taking the limit $n \to \infty$ and then as $\epsilon \to 0$ yields a contradiction.

Suppose that $\inf\limits_{x,y\in K} c\,(xy) > 0$. Then

$$\|x_n - Tx_n\| \leqslant \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\|$$
$$= \|Tx_{n-1} - RTx_n\| + \gamma_n$$
$$\leqslant \|RTx_{n-1} - Tx_n\| + \gamma_n$$
$$\leqslant a\|x_{n-1} - x_n\| + b\,\|x_{n-1} - Tx_{n-1}\| + b'\,\|x_n - Tx_n\|$$
$$+ c\,[\|x_{n-1} - Tx_n\| + \|x_n - Tx_{n-1}]] + \gamma_n.$$

where a, b, b', c are evaluated at (x_{n-1}, x_n)

$$(1 - b')\,\|x_n - Tx_n\| \leqslant a\,\|x_{n-1} - x_n\| + b\,\|x_{n-1} - Tx_{n-1}\|$$
$$+ c\eta\left[1 - \tfrac{1}{2}\delta\left[\frac{d'}{M}\right]\right](d + \epsilon) + (1 + c)\,\rho_n + c\gamma_{n-1}.$$

Since $d \leqslant \|x_n - Tx_n\|$,

$$d \leqslant (a + b + b' + 2c - 2\,(1 - \zeta)\,c)\,(d + \epsilon) + (1 + c)\,\gamma_n + c\gamma_{n-1},$$
$$\leqslant [\sup_{x,y\in K} (a + b + b' - 2c) - \inf_{x,y\in D} 2\,[1 - \zeta)\,c]\,(d + \epsilon)$$
$$+ \sup_{x,y\in K} (1 + c)\,\gamma_n + \sup_{x,p\in K} c\gamma_{n-1}.$$

Taking the limit as $n \to \infty$ we obtain

$$d \leqslant [1 - \inf_{x,y\in K} 2\,(1 - \zeta)\,c]\,(d + \epsilon).$$

Now let $\epsilon \to 0$ and obtain a contradiction. The following result appears in Massa and Roux[3].

Theorem M—Let X be a uniformly convex space, K a closed convex subset of X. $TK \to K$ satisfying (1), (2),

$$\sup_{x,y \in K} (b(x, y) + c(y, x)) < 1 \qquad \ldots(9)$$

and

$$\inf_{\|x\| \leqslant r} \|x - Tx\| = 0 \text{ for some } r < \infty. \qquad \ldots(10)$$

Then T has a fixed point in K.

Condition (4) implies (9), and (10) is true by Lemma 3. The proof of Theorem M does not depend on T being a selfmap of K. Applying Theorem M completes the proof of our Theorem.

Remarks : 1 Theorem 1.1 of Samanta[4] is a special case of the Theorem by assigning a and b to be constants, with $c = 0$.

2. The condition $\beta > 0$ should be added to the yhpotheses of Theorem 1.1 of Samanta[4] since the proof of Lemma 1.3 of that paper is not valid unless β is positive.

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CHARACTERIZATIONS OF $T_{1/2}$-SPACES USING GENERALIZED V-SETS

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In this note the authors investigate properties of generalized v-sets in a topological space and characterize the notion of $T_{1/2}$-spaces due to Levine[3] by using generalized v-sets. Finally, a new class of topological spaces is introduced.

1. INTRODUCTION

Levine[3] introduced the concept of generalized closed sets of a topological space and a class of topological spaces called $T_{1/2}$-spaces. The author[3] proves every T_1-space is $T_{1/2}$ and every $T_{1/2}$-space is T_0, although neitherim plication is reversible. Dunham[2] defined a new closure operator by using the generalized closed set and investigated a ne w topology and its properties. And, the class of $T_{1/2}$-spaces is characterized by the new topology (cf. Theorem 3.7 of Dunham[2]). In this note we characterize the class of $T_{1/2}$-spaces by using generalized v-set. And we introduce a new class of topological spaces named T^v-spaces.

2. PRELIMINARIES

Throughout this note, (X, τ) denotes a topological space with a topology τ on which no separation axioms are assumed unless explicitly stated. We recall some definitions and properties ot Maki[4]. The most of them can be quoted in the sequel.

Definition 2.1—In a topological space (X, τ), a subset B is a v-set (resp. Λ-set), if $B = B^v$ (resp. $B = B^\Lambda$), where $B^v = \cup \{F : F \subset B, X - F \in \tau\}$ and $B^\Lambda = \cap \{U : U \supset B, U \in \tau\}$[5,7].

This paper is dedicated to Professor Hiroshi Toda on his 60th birthday.

Then, in (2.5), (2.7) and Remark 2.8 of Maki[4], we have

(2.2) $(X - B)^{\Lambda} = X - B^{v}$ for every subset B.

(2.3) Let $\{B_j : j \in J\}$ be a family of subsets of X.

Then

$(\cup \{B_j : j \in J\})^{v} \supset \cup \{B_j^{v} : j \in J\}$ and $(\cup \{B_j : j \in J\})^{\Lambda} = \cup \{B_j^{\Lambda} : j \in J\}$.

Moreover we have

(2.4) $B^{v} \subset B \subset \mathrm{cl}(B)$, where $\mathrm{cl}(B)$ is the closure of B in (X, τ).

Definition 2.5—In a topological space (X, τ), a subset B is a generalized Λ-set[4] (abbreviated by g. Λ-set) if $B^{\Lambda} \subset F$ whenever $B \subset F$ and F is closed. A subset B is a generalized v-set[1] (abbreviated by g. v-set) of (X, τ) if $X - B$ is a g. Λ-set of (X, τ).

Definition 2.6—For a subset B of a topological space (X, τ), we define the following subsets :

$$c^{\Lambda}(B) = \cap \{U : B \subset U, U \in \mathscr{D}^{\Lambda}\} \text{ and } \mathrm{int}^{v}(B) = \cup \{F : B \supset F, F \in \mathscr{D}^{v}\},$$

where $\mathscr{D}^{\Lambda}$ (resp. $\mathscr{D}^{v}$) denotes the set of all g. Λ-sets (resp. g. v-sets) in (X, τ).

In Proposition 3.2, Example 3.3, Proposition 3.5 and Theorem 4.6 of Maki[4] we have the followings.

(2.7) Every Λ-set (resp. v-set) is a g. Λ-set (resp. g. v-set).

(2.8) Let J be an indexed set. If $B_j \in \mathscr{D}^{\Lambda}$ (resp. $B_j \in \mathscr{D}^{v}$) for all $j \equiv J$, then $\cup \{B_j : j \in J\} \in \mathscr{D}^{\Lambda}$ (resp. $\cap \{B_j : j \in J\} \in \mathscr{D}^{v}$}.

The intersection of two g. Λ-sets is not a g. Λ-set in general.

(2.9) For each $x \in X$, $\{x\}$ is an open set or a g. v-set.

(2.10) c^{Λ} is a Kuratowski closure operator on X.

Definition 2.11 (cf. 4.2) Definition 4.7 and Proposition 4.8 (i) of Maki[4])—Let τ^{Λ} be the topology on X generated by c^{Λ} in the usual manner, i. e., $\tau^{\Lambda} = \{B : B \subset X, c^{\Lambda}(X - B) = X - B.\}$ Then $\tau^{\Lambda} = \{B : B \subset X, \mathrm{int}^{v}(B) = B\}$ since $c^{\Lambda}(X - B) = X - \mathrm{int}^{v}(B)$ for every subset B of (X, τ).

In Theorem 4.8 (iii) of Maki[4] we have

(2.12) $\mathscr{F} \subset \tilde{\tau} \subset \mathscr{D}^{v} \subset \tau^{\Lambda}$, where $\tilde{\tau}$ (resp. $\mathscr{F}$) denotes the family of all v-sets (resp. closed sets) in (X, τ).

3. GENERALIZED V-SETS

In this section we characterize the notion of generalized v-sets of Definition 2.5 by using v-operations and we obtain results concerning such sets.

Proposition 3.1—A subset B of a topological space (X, τ) is a g. ν-set if and only if $U \subset B^{\vee}$ whenever $U \subset B$ and U is open.

PROOF : *Necessity*—Let U be an open subset of (X, τ) such that $U \subset B$. Then, since $X - U$ is closed and $X - U \supset X - B$, we have $X - U \supset X - B^{\nu}$ accordingto (2.2) and Definition 2 5.

Sufficiency—Let F be a closed subset of (X, τ) such that $X - B \subset F$. Since $X—F$ is open and $X - F \subset B$, by assumption we have $X - F \subset B^{\nu}$. Then, $F \supset (X - B)^{\wedge}$ by (2.2), and $X - B$ is a $g.\Lambda$-set (i. e., B is a $g.\nu$-set).

Proposition 3.2—Let B be a $g.\nu$-set in a topological space (X, τ). Then, for every closed set F such that

$$B^{\nu} \cup (X - B) \subset F, F = X \text{ holds.}$$

PROOF : The assumption $B^{\nu} \cup (X - B) \subset F$ implies $(X - B^{\nu}) \cap B \supset X - F$.

Since B is a g. ν-set, by Proposition 3.1 we have $B^{\nu} \supset X - F$ and hence $\phi = (X - B^{\nu}) \cap B^{\nu} \supset X - F$. Therefore we have $X = F$.

Corollary 3.3—Let B be a g. ν-set of (X, τ). $B^{\nu} \cup (X - B)$ is a closed set if and only if B is a ν-set.

PROOF : The proof of necessity is obtained by Proposition 3.2. The converse is obvious.

Remark 3.4: We give an example of a topological space which shows that the condition of Proposition 3.2 (i. e., B is a g. ν-set) cannot be removed. Let X be the set of all real numbers. On X, we can define the open sets of a topology to be ϕ and any subset of X that contains a particular point p, which is called a particular point topology[6], say τ. Let $B = \{x \in X : x \leqq p - 1\} \cup \{p\}$, then $B^{\nu} = B - \{p\}$. In (X, τ), B is not a g. ν-set, and $B^{\nu} \cup (X - B) = X - \{p\}$ holds and it is a closed set.

Proposition 3.5—Let B be a subset of topological space (X, τ) such that B^{ν} is closed. If $X = F$ holds for every closed subset F such that

$$F \supset B^{\nu} \cup (X - B), \text{ then } B \text{ is a } g.\nu\text{-set.}$$

PROOF : Let U be an open subset contained in B. According to assumption, $B^{\nu} \cup (X - U)$ is closed and it contains $B^{\nu} \cup (X - B)$. It follows that $B^{\nu} \cup (X - U) = X$ and hence $B^{\nu} \supset U$. Then B is a g. ν-set.

Remark 3.6 : Let X be a finite set, and let τ be a topology on X. In the topological space (X, τ) B^{ν} is closed for every subset B of X.

Remark 3.7: The following example shows that the condition of Proposition 3.5 (i. e., B^{ν} is a closed set) cannot be removed. Let X be the set $[-1, 1] = \{x \in R : -1 \leqq x \leqq 1\}$, and let τ be the overlapping interval topology[6] which is generated from sets of the form $[-1, b)$ for $b > 0$ and $(a, 1]$ for $a < 0$. Let $B = [-1, 1/2)$. Then

$B^{v} = [-1, 0)$ is not closed. Any closed set containing $B^{v} \cup \{X - B\}$ is X, however B is not a g. v-set.

4. Characterizations of $T_{1/2}$-spaces

In a topological space (X, τ) a subset B of X is said to be g. closed[3] if $\mathrm{cl}\,(B) \subset U$ whenever $B \subset U$ and U is open.

Definition 4.1[3]—A topological space (X, τ) is said to be a $T_{1/2}$-space, if every g. closed set is closed.

The notion of g. v-sets in a topological space (X, τ) does not coincide with the one of g. closed sets even if (X, τ) is a finite topological space (cf. Remark 4.10 of Maki[4]). However the class of $T_{1/2}$-spaces is characterized by the notion of $g.v$-sets (cf.. Theorem 3.7 of Dunham[2]). In the following theorem we give the characterization of the class of $T_{1/2}$-spaces.

Theorem 4.2—The statements about a topological space (X, τ) are equivalent.

(i) (X, τ) is a $T_{1/2}$-space.

(ii) Every g. v-set is a v-set.

(iii) Every $\tau^{\wedge}$-open set is a v-set.

PROOF : (i) $\rightarrow$ (ii) Supposed that there exists a g.v-set B which is not a v-set. Since $B^{v} \subsetneqq B$ there exists a point x of B such that $x \notin B^{v}$. Then the singleton $\{x\}$ is not closed. According to Theorem 2.2 of Dunham[2] the complement of $\{x\}$ (i. e., $X - \{x\}$) is a g. closed set. On the other hand, we have $\{x\}$ is not open since B is a g. v-set and $x \notin B^{v}$. Therefore we have $X - \{x\}$ is not closed but it is a g. closed. This contradicts to the assumption that (X, τ) is a $T_{1}/_{2}$-space.

(ii) $\rightarrow$ (i) Suppose that (X, τ) is not a $T_{1/2}$-space. Then, there exists a g. closed set B which is not closed. Since B is not closed, there exists a point x such that $x \notin B$ and $x \in \mathrm{cl}\,(B)$. In (2.9) we have the singleton $\{x\}$ is an open set or it is a g. v-set. When $\{x\}$ is open, we have $\{x\} \cap B \neq \phi$ because $x \in \mathrm{cl}\,(B)$. This is a contradiction. Let us consider the case : $\{x\}$ is a g. v-set. If $\{x\}$ is not closed, we have $\{x\}^{v} = \phi$ and hence $\{x\}$ is not a v-set. This contradicts to (ii). Next, if $\{x\}$ is closed, we have $X - \{x\} \supset \mathrm{cl}\,(B)$ (i. e., $x \notin \mathrm{cl}\,(B)$). In fact, the open set $X - \{x\}$ contains the set B which is a g. closed set. Then, this also contradicts to the fact that $x \in \mathrm{cl}\,(B)$. Therefore (X, τ) is a $T_{1/2}$-space.

(ii) $\rightarrow$ (iii) Let B be a $\tau^{\wedge}$-open set, i. e., $B = \mathrm{int}^{v}\,(B)$. We claim that $\mathrm{int}^{v}\,(B)$ is v-set under given assumptions. Then B is a v-set. To prove the claim we note that $\tilde{\tau} = \mathscr{D}^{v}$ according to the assumption (ii) and (2.7). By using (2.3) and the fact that $\mathscr{D}^{v} = \tilde{\tau}$ we have $(\mathrm{int}^{v}\,(B))^{v} = (\cup \{F : B \supset F,\ F \in \tilde{\tau}\})^{v} \supset \cup \{F^{v} : B \supset F,\ F \in \tilde{\tau}\} = \mathrm{int}^{v}\,(B)$. Then, by (2.4), we have $(\mathrm{int}^{v}\,(B))^{v} = \mathrm{int}^{v}\,(B)$, and hence $\mathrm{int}^{v}\,(B)$ is a v-set.

(iii) $\rightarrow$ (ii) Let B be a g. v-set. Then $\mathrm{int}^{v}\,(B) = B$ by Definition 2.6. It follows from the fact in Definition 2.11 that $B \in \tau^{\wedge}$. Hence B is a v-set, by (iii).

Corollary 4.3 (Dunham[2], Theorem 2.6)—A topological space (X, τ) is $T_{1/2}$ if and only if every singleton in (X, τ) is open or closed.

Proof : *Necessity*—Let $x \in X$. If $\{x\}$ is not open, then it is a $g.\nu$-set by (2.9). it follows from Theorem 4.2 and assumption that $\{x\}$ is a ν-set. Since $\{y\}^\nu = \phi$ for every non closed set $\{y\}$, we have $\{x\}$ is a closed set. Therefore $\{x\}$ is open or closed.

Sufficiency—Suppose that (X, τ) is not a $T_{1/2}$-space. Then, there exists a $g.\nu$-set B which is not a ν-set. Then there exists a point $x \in X$ such that $x \in B$ and $x \notin B^\nu$. If $\{x\}$ is closed, B^ν contains the closed set $\{x\}$. This is impossible. If $\{x\}$ is open, the closed set $X - \{x\}$ contains $B^\nu \cup (X - B)$. By using Proposition 3.2 we have $X - \{x\} = X$ which is impossible. Therefore (X, τ) is a $T_{1/2}$-space.

Corollary 4.4—(i) For any topological space (X, τ), (X, τ^Λ) is a $T_{1/2}$-space.

(ii) For any topology τ, $(\tau^\Lambda)^\Lambda = (\tau^\Lambda),^\sim$ where $(\tau^\Lambda)^\sim$ is the family of all ν-sets in the topological space (X, τ^Λ).

Proof : (i) Let $x \in X$. If $\{x\}$ is open (i. e., $\{x\} \in \tau$), then $X - \{x\} \in \tau^\Lambda$ (i.e., $\{x\}$ is τ^Λ-closed), by (2.12). If $\{x\}$ is not open then $\{x\}$ is a $g.\nu$-set by (2.9). Then by (2.12) we have $\{x\} \in \tau^\Lambda$ (i. e., $\{x\}$ is τ^Λ-open). Therefore every singleton in X is open or closed in (X, τ^Λ). By Corollary 4.3 we obtain that (X, τ^Λ) is a $T_{1/2}$-space.

(ii) It follows from (i) and Theorem 4.2 that $(\tau^\Lambda)^\Lambda \subset (\tau^\pi)^\sim$. By using (2.12) we obtain the equality.

5. Comparisons

We define a class of topological spaces called T^ν-spaces as an analogy to $T_{1/2}$-spaces, and we investigate relations between T^ν-spaces and several spaces.

Definition 5.1—A topological space (X, τ) is said to be a T^ν-space, if every τ^Λ-open set is a $g.$ ν-set (i. e., $\mathscr{D}^\nu = \tau^\Lambda$).

Lemma 5.2—For a topological space (X, τ), every singleton of X is $g.\Lambda$-set if and only if $G = G^\nu$ holds tor every open set G.

Proof : *Necessity* —Let G be an open set. Let $y \in X - G$, then $\{y\}^\Lambda \subset X-G$ by assumption. By using (2.3) we have $X - G \supset \cup \{\{y\}^\Lambda : y \in X - G\} = (X - G)^\Lambda$, and hence $X - G = (X - G)^\Lambda$. Then it follows from (2.2) that $G = G^\nu$.

Sufficiency—Let $x \in X$ and F be a closed set such that $\{x\} \subset F$. Since $X - F = (X - F)^\nu = X - F^\Lambda$, we have $F = F^\Lambda$. Therefore we have $\{x\}^\Lambda \subset F^\Lambda = F$. Hence $\{x\}$ is a $g.\Lambda$-set.

Proposition 5.3—For a topological space (X, τ), we have the following implications.

(i) If (X, τ) is a $T_{1/2}$-space then it is a T^ν-space.

(ii) If (X, τ) is an R_0-space then it is a T^v--space.

PROOF : (i) In (2.12) we have $\tilde{\tau} \subset \mathscr{D}^v \subset \tau^\Lambda$. By using Theorem 4.2 we have $\tilde{\tau} = \tau^\Lambda$. Therefore we obtain $\tau^\Lambda = \mathscr{D}^v$. Hence, (X, τ) is a T^v-space.

(ii) To prove (ii) we recall the following fact. In Theorem 2.2 of Dube[1], it is shown that (X, τ) is an R_0-space if and only if any open set G in (X, τ) can be expressed as $G = \cup \{F : F \subset G, X - F \in \tau\}$, i. e., $G = G^v$ (cf. Definition 2.1). Let B be a subset of (X, τ). It follows from Lemma 5.2 and (2.8) that $B = \cup \{\{b\} : b \in B\}$ is a $g.\Lambda$-set. Thus we have every subset of (τ X, τ) is a $g.$ v-set, and hence $\mathscr{D}^v$ coincides with the discrete topology of X. By using (2.12) we obtain $\mathscr{D}^v = \tau^\Lambda$. Hence, (X, τ) is a T^v-space.

The following examples show that none of these implications is reversible.

Example 5.4—Let $X = \{a, b, c\}$ and let $\tau = \{\phi, \{a\}, \{b, c\}, X\}$. The space (X,τ) is not a T_0-space. Since $\{a\}^v = \{a, b\}^v = \{a, c\}^v = \{a\}$, $\{b\}^v = \{c\}^v = \phi$ and $\{b, c)^v = \{b, c\}$, we have $\mathscr{D}^v$ coincides with the discrete topology of X and hence $\mathscr{D}^v = \tau^\Lambda$. These imply that (X, τ) is not $T_{1/2}$ but T^v.

Example 5.5—Let $X = \{a, b, c\}$ and let $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. Then the space (X, τ) is not R_0 but T^v. In fact, we have $\mathscr{D}^v = \{\phi, \{b\}, \{c\}, \{b, c\}, X\} = \tau^\Lambda$ since $\{a\}^v = \{b\}^v = \{a, b\}^v = \phi$, $\{a, c\}^v = \{c\}$, $\{c\}^v = \{c\}$ and $\{b, c\}^v = \{b, c\}$. Thus, (X, τ) is not a T^v-space. Since $\{a\}^v \neq \{a\}$ for an open set $\{a\}$, (X, τ) is not an R_0-space.

T_s is independent of T_0 an Example 5.4 and the following example show.

Example 5.6—Let X be the set of all real numbers R and let $\tau = \{\phi, X\} \cup \{\{a, \infty\} : a \in X\}$, where $\{a, \infty\} = \{x \in X : a < x\}$. Then the spaee (X, τ) is not T^v but T_0. In fact, we obtain that τ^Λ coincides with the discrete topology of R however $\{a\} \cup \{a, \infty\}$ is not a $g.$ v-set.

From the results in this action we obtain the following diagram :

$$\begin{tikzcd}
T_{1/2} \arrow[r, shift left] & T^v \arrow[l, shift left, "/" marking] \arrow[r, shift right, "/" marking] \arrow[d, shift left, "/" marking] & R_0 \arrow[l, shift right] \\
& T_0 \arrow[u, shift left, "/" marking] &
\end{tikzcd}$$

where $A \rightarrow B$ (resp. $A \not\rightarrow B$) represents that A implies B (resp. A does not always imply B).

In the following proposition we have a further result concerning the transfer of properties from (X, τ) to (X, τ^Λ) (cf. Corollary 4.4 (i)).

Proposition 5.7—If (X, τ) is an R_0-space then (X, τ^Λ) is T_1-space.

PROOF : By using Theorem 2.2 of Dube[1] and Lemma 5.2 that every singleton $\{x\}$ of X is a g. Λ-set. Then we have $c^{\Lambda}(\{x\}) = \{x\}$ and hence $\{x\}$ is a τ^{Λ}-closed set. Therefore every singleton is closed in (X, τ^{Λ}).

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GOLDIE MODULES

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A Goldie module is defined as a finite dimensional module M over a ring R satisfying the a. c.c. for annihilators of subsets of M in R. Here we establish some results analogous to those of a Noetherian module viz. decomposition of zero of a Goldie module and the Artin Rees theorem for a Goldie module in some special cases.

The attached ring here need not be a Noetherian ring in general as assumed in some corresponding results in a Noetherian module.

1. INTRODUCTION

The purpose of this paper is to introduce the notion of a Goldie module and to establish some elementary properties of such a system

Throughout this paper, unless otherwise specified M will be a left unitary module over the ring R with unity.

A left module M over a ring R is a 'left Goldie module over R or a left Goldie R-module if'

(1) M is finte dimensional

(2) R satisfies the ascending chain

condition (a. c. c.) for annihilators of subsets of M, i. e., any collection of annihilators of subsets of M in R satisfies the maximal condition. Similarly we have a 'right Goldie module'. We shall confine ourselves to left Goldie modules only.

We know that a left Goldie ring is a ring R such that R is finite dimensional as a left R-module and such that the left annihilator ideals in R satisfy the a. c. c. Clearly every left Goldie ring is a left Goldie module over itself. Also, every finite dimensional left module (e. g. a Noetherian module) over a left Noetherian ring R is a left Goldie module over R. Moreover it can easily be seen that if M is an Artinian left module over a left Artinian ring R, then M is a left Goldie module over R. And also if M is an (R, S) bimodule such that M is Noetherian as an R-module and Artinian as an S-module, then M is a left Goldie module over R. It may be noted that a sub-module of a Goldie module is again Goldie.

It is known that a homomorphic image of a left Goldie ring need not be a left Goldie ring (page 4 of Jategaonkar[4]). And it is easily seen that a quotient ring R/I is left Goldie if and only if the left R-module R/I is a left Goldie module. Thus homomorphic image of a Goldie module need not be a Goldie module. And this prompts us to make the following definitions (as in case of Goldie ring).

If every homomorphic image of a left Goldie module M is a left Goldie module, then M is a 'fully left Goldie module'.

If for any closed submodule N of a left Goldie module M, the quotient module M/N is left Goldie then M is a 'fully Goldie module with respect to closed sudmodules'.

Let M be a Goldie module with closed submodules $M_1, \ldots, M_t$ such that

(1) $M \cap M_2 \cap \ldots \cap M_t = (0)$, but

$M_1 \cap \ldots \cap \hat{M_i} \cap \ldots \cap M_t \neq (0)$, for any $i = 1, \ldots, t$, where $\wedge$ denotes omission of the symbol underneath it,

(2) each quotient module M/M_i is a primary submodule with

$$A\,(M/M_i) \neq A\,(M/M_j) \text{ for } i \neq j$$

where $A\,(X)$ drnotes the associated primes of X for an R-module X.

Then $M_1 \cap \ldots \cap M_t$ is a primary decomposition of O in M. We prove in section 3 the existence of a primary decomposition of O in a Goldie module with certain conditions.

It can be easily seen that the direct sum of two Goldie modules is again a Goldie module. Next we prove that the set of associated primes of a module M which is fully Goldie with respect to closed submodules and where any prime submodule N/M' of a quotient M/M' of it has a prime closed extension T/M' such that T is a closed extension of M', is finite. Another interesting result we are going to prove is that the set of associated primes of a commutative semiprime Goldie ring, when considered as a Goldie module, is the set of minimal prime ideals of the ring R, which gives as an obvious corollary that $\bigcap_{P \in A(R)} P = (0)$. Then we show the existence of a primary decomposition of O in M where M is a Goldie module. Moreover, if $M_1 \cap M_2 \cap \ldots \cap M_t$ is a primary decomposition then

$A\,(M) = A\,(M/M_1) \cup \ldots \cup A(M/M_t)$. Two other important results we prove are the following :

(i) Let the module M be a nonsingular Goldie module over a commutative Noetherian ring R. Then $\sqrt{l(M)} = \bigcap_{P \in A(M)} P$ where $\sqrt{l(M)}$ is the nil radical of the left annihilator $l\,(M)$ of M in R.

(ii) Let M be as above and is such that all its quotient modules are nonsingular over R. Then given any submodule N of M, any ideal B of R, any $n \in Z^+$, there exists a submodule M' of M such that

$$B^n \subseteq \sqrt{l\,(M/M')} \text{ and } B^n N = M' \cap N.$$

Finally we prove the Goldie analogue of Artin-Rees theorem[1]. Let M be a module described as in the preceding result. Then for any $n \in Z^+$, there exists $at \in Z^+$ such that $B^{nt}M \cap N \subseteq B^n N$.

2. Preliminaries

We now give some preliminary lemmas for use in the proofs of the main results in section 3.

A submodule K of a module M is proper essential extension in M if there is a sub-module N of K such that $N \leqslant_e K$ (N is an essential submodule of K) and $N \neq K$.

Lemma 2.1 – If M is a module, $N \leqslant A$ and $A \leqslant {}_cM$ (A is a closed submodule of M), then $A/N \leqslant {}_cM/N$.

PROOF : The result follows from (Proposition 1.1C of Goodearl[3]).

Lemma 2.2—If M is a left Goldie module over a ring R, then $A\,(M) = \phi$ if and only if $M = (0)$.

PROOF : $M = (0)$ implies $A\,(M) = \phi$, for then it has no proper submodule.

Conversely, let $M \neq (0)$ and

$$F = \{l\,(M^*) \mid M^* \leqslant M,\ M^* \neq (0)\}.$$

Then F is a set of left annihilator ideals of R. Since M is left Goldie, F has a maximal element, say $l\,(N)$, where N is a nonzero submodule of M. It $N' \leqslant N$, $N' \neq (0)$ then $l\,(N') \supseteq l\,(N)$ and since $l\,(N)$ is maximal in F, $l\,(N') = l\,(N)$. Thus N is a prime submodule of M. Hence $l\,(N) \in A\,(M)$ giving $A\,(M) \neq \phi$.

Lemma 2.3—If M is a left Goldie module with an exact sequence

$$(0) \longrightarrow M' \xrightarrow{g} M \xrightarrow{f} M'' \longrightarrow (0)$$

then the associated primes of M, M' and M'' are connected by the inclusions

$$A\,(M\,) \subseteq A\,(M) \subseteq A\,(M') \cup A\,(M'').$$

PROOF : If $M = (0)$ the result follows trivially. If $M \neq (0)$, then by Lemma 2.2 $A\,(M) \neq \phi$. The module M' can be considered as a sub-module of M, since g is injective. Thus $A\,(M') \subseteq A\,(M)$.

Now let $l\,(N) \in A\,(M)$. Then N is a prime submodule of M.

Suppose $M' \cap N \neq (0)$. Then $l(N) = l(M' \cap N)$, since N is a prime submodule of M. And $M' \cap N$, being a submodule of the prime submodule N, is also prime. Therefore $l(M' \cap N) \in A(M')$. Thus $l(N) \in A(M')$.

Next suppose $M' \cap N = (0)$. If h is the restriction of f to N, then h is injective. So we can consider N as a submodule of M''. It follows that $l(N) \in A(M'')$ giving $A(M) \subseteq A(M'')$. Therefore $A(M) \subseteq A(M') \cup A(M'')$.

Now we consider the Goldie module M such that any prime submodule N/M' of any quotient M/M' has a prime closed extension T/M' such that $M' \leqslant N \leqslant T \leqslant_c M$.

As a modules the Goldie rings Z_3, Z_6, Z_{15} are such that their quotients are all prime. And in a prime module all of its submodules are primes and at least the module itself is a prime closed extension of each of its prime submodules. Again $M = Z_{30}$ is such a Goldie module (over Z_{30} or Z) that its submodules are $A_1 = \{0, 2, 4, \ldots, 28\}$

$A_2 = \{0, 3, 6, \ldots, 27\}$, $A_3 = \{0, 5, 10, \ldots, 25\}$, $A_4 = \{0, 6, 12, \ldots, 24\}$,

$A_5 = \{0, 10, 20\}$ and $A_6 = \{0, 15\}$ of which $A_4 < A_2 <_c M$,

$A_5 < A_3 <_c M$ and $A_6 < A_3 <_c M$. So by Lemma 2.1, $A_2/A_4 <_c M/A_4$,

$A_3/A_5 <_c M/A_5$ and $A_3/A_6 <_c M/A_6$. Here each of A_2/A_4, A_3/A_5 and A_3/A_6 is prime submodule. Therefore each is a closed extension of itself which is prime. And the remaining quotients M/A_1, M/A_2 and M/A_3 are all primes.

So all these Goldie modules satisfy the assumed property that any prime submodule N/M' of M/M' has a prime closed extension T/M' such that $M \leqslant N \leqslant T \leqslant_c M$.

Moreover we note that the above modules are such that all of its quotient modules are nonsingular as R-modules.

The following important lemma leads us to the main results.

Lemma 2.4—Let M be a left module which is fully Goldie with respect to closed submodules and is such that any prime submodule N/M' of a quotient M/M' of it has a prime closed extension T/M' such that T is a closed extension of M'. If the set $A(M)$ is expressed as a union of two disjoint sets X and Y, then there exists an $M' \leqslant_c M$ such that $A(M') = X$, $A(M/M') = Y$ and $A(M) = A(M') \cup A(M/M')$.

Proof : Let $F = \{N \leqslant_c M \mid A(N) \leqslant X\}$. Since O is a closed submodule and $A(0) = \phi$ (by Lemma 2.1), we have $F \neq \phi$. And since M is Goldie it is finite dimensional and so it satisfies the a.c.c. on closed submodules (Theorem 3.14 of Goodearl[3]). Thus F has a maximal element, say M'. On the other hand $X \cup A(M/M') \supseteq X \cup Y$. Since $X \cap Y = \phi$, it follows that $A(M/M') \supseteq Y$. If possible let $A(M/M^1) \nsubseteq Y$. Then there exists a prime submodule N/M' of M/M' such that $l(N/M) \in A(M/M)$ and $\notin Y$. By hypothesis there is a prime extension T/M' such that $N/M' \leqslant T/M' \leqslant_c M/M'$ and $M' \leqslant N \leqslant T \leqslant_c M$.

Since T/M' is nonzero, $M' \subset T$, and $l(T/M') = l(N/M')$. Let us write P for $\{l(N/M)\}$. Then $A(T/M') = P$ and $P \nsubseteq Y$.

From the exact sequence

$$(0) \longrightarrow M' \longrightarrow T \longrightarrow T/M' \longrightarrow (0).$$

We get $A(T) \subseteq A(M') \cup A(T/M')$ by Lemma 2.2. Since $A(M') \subseteq X$ and $A(T/M') = P$, we get $A(T) \subseteq X \cup P$. On the other hand $T \leqslant M$ and $A(M) = X \cup Y$ give $A(T) \subseteq X \cup Y$. Since $A(T) \subseteq X \cup P$ and $A(T) \subseteq X \cup Y$, $P \not\subset Y$, $A(T) \subset X$. And T is closed in M and hence $T \in F$. Since $M' \subseteq T$, this contradicts the maximality of M' in F. It follows that $A(M/M') \subseteq Y$ so that $A(M/M') = Y$. Now from the inclusion relation

$$A(M) \subseteq A(M') \cup A(M/M').$$

it follows that $X \cup Y \subseteq A(M') \cup Y$, which gives $X \subseteq A(M')$, since $X \cap Y = \phi$. But since M' is in F, $A(M') \subseteq X$ and hence

$$A(M) = X \cup Y = A(M') \cup A(M/M')$$

Lemma 2.5—If M_1 and M_2 are two modules over a ring, then $A(M_1 \oplus M_2) = A(M_1) \cup A(M_2)$.

PROOF : An arbitrary element of $A(M_1 \oplus M_2)$ is $l(Q)$, where Q is a prime submodule of $M_1 \oplus M_2$.

Now we may assume $Q = Q_1 \oplus Q_2$ where $Q_1 \leqslant M_1$, $Q_2 \leqslant M_2$ and each of Q_1 and Q_2 is prime. Therefore $l(Q_1) = l(Q_2) = l(Q)$. Thus

$$A(M_1 \oplus M_2) \subseteq A(M_1) \cup A(M_2)$$

Since a prime submodule of Q_1 (Q_2) is a prime submodule of $M_1 \oplus M_2$, the opposite inclusion follows.

Lemma 2.6—If M is a left module and P, Q, N are submodules of M such that $N \leqslant P, Q$ then

$$A(P \cap Q/N) = A(P/N \cap Q/N) \subseteq A(P/N) \cap A(Q/N).$$

PROOF : The result follows from the following observations : A prime submodule of $P \cap Q/N$ is also a prime submodule of P/N and Q/N and conversely.

A prime submodule of $P/N \cap Q/N$ is prime submodule of P/N and Q/N.

We assume $\bigcap_{P \in A(M)} P \neq (0)$ in general and write $\mathscr{P} = \bigcap_{P \in A(M)} P$.

Lemma 2.7—Let M be a non singular Goldie module over a finite dimensional ring R. Then for any $x \in \bigcap_{P \in A(M)} P$, there exists a $t \in Z^+$ such that $x^t \in l(M)$.

PROOF : Let $x \in \bigcap_{P \in A(M)} P$. Then for every $P \in A(M)$ we get module homomorphism $\phi_i : M \to M$, $m \to x^i m$, $i = 1, 2, \ldots$ Clearly Ker $\phi_i \subseteq$ Ker ϕ_{i+1}.

In other words $r_M(x^i) \subseteq r_M(x^{i+1})$ which gives $l_R(r_M(x^i)) \supseteq l_R(r_M(x^{i+1}))$.

If $X \subseteq Y \subseteq M$, then we get $l_R(X) \supseteq l_R(Y)$.

Let $A = l_R(Y)$, $B = l_R(X)$, Now, if A is an essential R-submodule of B there is an essential left ideal L of R such that $lb \subseteq A$ (of Lemma 1.1 Chatters and Hazaranavis[2]). Hence $lb\, r_M(A) = 0$ which gives $br_M(A) = 0$, for M is non singular left R-module. Thus $b \in l_R(r_M(A))$. Therefore $b \in A\ (= l_R(r_M(l_R(Y)))$. Hence $A = B$. So if A is strictly contained in B, then there is a nonzero ideal C of R such that $C \subseteq B$ and $A \cap C = 0$. It follows that a strictly descending chain of left ideals of above type gives rise to a direct sum of nonzero ideals such as C. And since R is finite dimensional as a module, the above chain must stop. So $l_R(r_M(x^t)) = l_R(x^{t+1}))$ for some t which gives $r_M(l_R(r_M(x^t))) = r_M(l_M(r_M(x^{t+1})))$, i. e. $r_M(x^t) = r_M(x^{t+1})$.

Consider the homorphism

$$f : x^t M \to x^t M,\ x^t m \to x^{t+1}\ m.$$

If as elements of $x^t M$,

$$x^{t+1} m = x^{t+1} n$$

then

$$x^{t+1}(m - n) = 0$$

$$\Rightarrow m - n \in \ker \phi_{t+1} = \ker \phi_t$$

$$\Rightarrow x^t (m - n) = 0$$

$$\Rightarrow x^t m = x^t n.$$

Hence f is injective, Therefore $x^t M \leqslant M$ and this gives $A(x^t M) \subseteq A(M)$. If $x^t M \neq (0)$, then $A(x^t M) \neq \phi$. Hence there exists at least one nonzero submodule N' of $x^t M$ such that $l(N') \in A(x^t M)$. Since $x \in P$ for every $P \in A(M)$, we get $x \in P$ for every $P \in A(x^t M)$. So $x \in l(N')$ which implies $xN' = (0)$ i. e. $f(N') = 0$ and since f is injective, it follows that $N' = (0)$, a contradiction. Hence $x^t M = (0)$ i. e., $x^t \in l(M)$.

Lemma 2.8—Let M be a fully Goldie module over a commutative ring R such that any prime submodule N/M' of a quotient M/M' of it has a prime closed extension T/M' such that $M' \leqslant N \leqslant T \leqslant_c M$. Then given any submodule N of M, any ideal B of R and any positive integer n, there is a submodule $B^n N$ of N satisfying the condition :

$$B^n N = M' \cap N \text{ and } A(N/B^n N) = A(M/M').$$

PROOF : We note that every quotient module $(M/B^n N)/(D/B^n N)$ is isomorphic to M/D which is Goldie, and any prime submodule of a quotient of $M/B^n N$ has a prime

closed extension. Thus by Lemma 2.4 if $A(N/B^nN) = Y$, $A(M/B^nN) = X \cup Y$ and $X \cap Y = \phi$, then there is a submodule M' of M containing B^nN, with an exact sequence

$$(0) \to M'/B^nN \to M/B^nN \to M/M' \; (= (M/B^nN)/(M'/B^nN)) \to (0)$$

such that $A(M'/B^nN) = X$ and $A(M/M') = Y$. We then get $A(N/B^nN)$ $(= Y) = A(M/M')$, as stated. And by Lemma 2.6.

$$A(M' \cap N/B^nN) = A(M'/B^nN \cap N/B^nN)$$
$$\subseteq A(M'/B^nN) \cap A(N/B^nN).$$

Thus, $A(M' \cap N/B^nN) = \phi$, since $X \cap Y = \phi$. And $M' \cap N/B^nN$, being a submodule of the Goldie module M/B^nN, is Goldie. Therefore by Lemma 2.2, $A(M' \cap N/B^nN) = \phi$ implies $M' \cap N \subseteq B^nN$.

Moreover, $B^nN \subseteq N$ and M' contains the submodule B^nN. Hence $B^nN \subseteq M' \cap N$. Thus $B^nN = M' \cap N$.

3. Main Results

We are now ready to present the main results of this paper.

Theorem 3.1—Let M be a left module which is fully Goldie with respect to closed submodules and is such that any prime submodule N/M' of a quotient M/M' has a prime closed extension T/M' such that $M' \leqslant N \leqslant T_c \leqslant M$. Then $A(M)$ is finite.

Proof : If possible let $A(M)$ be an infinite set viz., $\{P, Q, R, \ldots\}$. Write $\{P\}$ simply as P and $\{Q\}$ as Q etc.

Now if $A(M) = P \cup Y$ and $P \cap Y = \phi$, then by Lemma 2.4, there exists a closed submodule M' of M such that $A(M') = P$, $A(M/M) = Y$. And $A(M) = A(M') \cup A(M/M')$.

Since $Q \in A(M)$, we get $Q \in P \cup A(M/M')$ which gives $Q \in A(M/M')$.

Therefore there is one prime submodule B'/M' of M/M' such that $l(B'/M') = Q$. It follows that $A(B'/M') = Q$.

And by hypothesis there is one prime extension M''/M' such that

$$B/M' \leqslant M''/M' \leqslant_c M/M' \text{ and } M' \leqslant B' \leqslant M'' \leqslant_c M.$$

Hence

$$l(M''/M') = l(B'/M') = Q.$$

Therefore

$$A(M''/M) = A(B'/M') = Q.$$

Since

$$(0) \to M' \to M'' \to M''/M' \to (0)$$

is exact by Lemma 2.2, we get

$$A(M'') \subseteq A(M') \cup A(M''/M).$$

It follows that $A(M'') \subseteq \{P, Q\}$.

Again since the sequence

$$(0) \to M'' \to M \to M/M'' \to (0)$$

is exact, we get

$$A(M) \subseteq A(M'') \cup A(M/M'').$$

Therefore $A(M) \subseteq \{P, Q\} \cup A(M/M')$.

i. e.

$$R \in A(M/M').$$

And in a like manner we get another closed submodule M''' of M such that $M' \underset{\neq}{<} M'' \underset{\neq}{<} M''$ and for $S \in A(M)$ we get $S \in A(M/M'')$. Since $A(M)$ is infinite, we get a strictly ascending infinite sequence of closed submodules of M. But M, being Goldie is finite dimensional and therefore satisfies the a. c. c. on closed submodules. The contradiction shows that $A(M)$ is finite.

Theorem 3.2—Let R be a commutative semiprime Goldie ring with 1. If R is considered as a Goldie module over itself, then

$A(R)$ = the set of minimal prime ideals of the ring R.

PROOF : Let $P \in A(R)$. Then P is a prime ideal of R and $P = r(N)$ for some prime submodule N of R. We now show that P is a minimal prime ideal of R.

Let P' be any prime ideal such that $P' \subseteq P$. Now $(l(P))^2 \subseteq l(P)P' \subseteq l(P)P = 0$, if $l(P) \subseteq P'$, i. e., if $l(P) \subseteq P'$, then $l(P) = 0$. Since R is semiprime. Moreover $l(R) = 0$. Therefore $l(P) = l(R)$, which gives $(r\, l(P)) = r(l(R))$. Since $P = r(N)$ and $R = r(0)$, we get $r(l(r(N))) = r(l(r(0)))$, which gives $r(N) = r(0)$, i. e. $P = R$. And since P is prime, this is not possible. Therefore $l(P) \nsubseteq P'$. But P' is prime and $l(P)P = (0) \subseteq P'$. Hence $P \subseteq P'$ so that $P' = P$.

Conversely let P be a minimal prime ideal of R. Then $P = l(N)$ for some $N \subseteq R$ (Lemma 1.16 of Chatters and Hazarnavis[7]). Therefore $P = l(r(l(N))) = l(r(P))$. We claim that $r(P)$ is a prime submodule of R. Let I be a nonzero submodule of the R-module $r(P)$. Then $PI = (0)$. Hence $P \subseteq l(I)$. Since $I \subseteq P$ implies $I^2 (\subseteq IP) = (0)$, for R is semiprime, we get $I \nsubseteq P$. Therefore $\alpha \in l(I)$ gives $R\alpha \subseteq P$. Hence $\alpha \in P$, i. e., $l(I) \subseteq P$. Thus $P = l(I)$. So $l(I) = l(r(P))$, which implies that $r(P)$ is a prime submodule of R. Therefore $P = l(r(P)) \in A(R)$, and hence $A(R)$ = the set of minimal prime ideals of R.

Theorem 3.3—Let the module M be as in Lemma 2.7. Then

$$\sqrt{l(M)} = \bigcap_{P \in A(M)} P.$$

Proof : Let $r \in \sqrt{l(M)}$ and $P \in A(M)$. Then, for some $n \in Z^*$, we have $r^n \in l(M)$ and $P = l(N)$ for some prime submodule of M. (Hence, $r^n \in l(N) = P$). Therefore $r^n N = 0$ or, $r \in l(r^{n-1} N)$. Since $r^{n-1}N$ is a submodule of the prime submodule N, we get

$$l(r^{n-1}N) = l(N) = P.$$

Thus $r \in P$, i. e., $\sqrt{l(M)} \subseteq P$, $P \in A(M)$, which gives

$\sqrt{l(M)} \subseteq \bigcap_{P \in A(M)} P$. The opposite inclusion follows trivially from Lemma 2.7.

Theorem 3.4—Let the module M be as in Theorem 3.1. Then the following two results hold :

(I) There exists a primary decomposition of O in M.

(II) And if $M_1 \cap \ldots \cap M_t$ is a primary decomposition of O in M, then $A(M) = A(M/M_1) \cup \ldots \cup A(M/M_t)$.

Proof : (I) By Theorem 3.1 $A(M)$ is finite. Let $A(M) = \{P_1, \ldots, P_t\}$. Since $A(M)$ is expressible as the union of two disjoint sets

$\{P_1, \ldots, \hat{P}_i, \ldots, P_t\}$ and $\{P_i\}$ by Lemma 2.4 we get closed submodules $M_1, \ldots, M_t$ of M such that for every i we have

$$A(M_i) = \{P_1, \ldots, \hat{P}_i \ldots, P_t\}$$

and

$$A(M/M_i) = \{P_i\}.$$

Now for each i, M/M_i is primary and $A(M/M_i) \neq A(M/M_j)$ for $i \neq j$.

And,

$$A(M_1 \cap \ldots \cap M_t) \subseteq A(M_1) \cap \ldots \cap A(M_t).$$

Since clearly $A(M_1) \cap \ldots \cap A(M_t) = \phi$, we then have

$A(M_1 \cap \ldots \cap M_t) = \phi$, and therefore by Lemma 2.2,

$$M_1 \cap \ldots \cap M_t = (0)$$

If possible let $M_1 \cap \ldots \cap \hat{M}_i \cap \ldots \cap M_t = (0)$,

i. e,

$$\bigcap_{j \neq i} M_j = (0),$$

for some i, $1 \leqslant i \leqslant t$. Then we get a homomorphism

$$\alpha : M \to \bigoplus_{j \neq i} M/M_j$$

$$m \to (m + M_1, \ldots, \widehat{m+M_i} \ldots, m + M_t)$$

We note that $\ker \alpha = \{m \mid m \in \bigcap_{j \neq i} M_j\}$

$$= (0), \text{ by our assumption.}$$

Therefore α is an embedding and hence $A(M) \subseteq A(\bigoplus_{j \neq i} M/M_j)$

It follows from Lemma 2.5 that for each i, $A(M) \subseteq \bigcup_{j=1} A(M/M_j)$

i. e., $A(M) \subseteq \{P_1, \ldots, \hat{P}_i, \ldots P_t\}$which is absurd. Hence $\bigcap_{j \neq i}^{t} M_j \neq (0)$.

(II) Next suppose that $\bigcap_{j=1}^{t} M_j$ is a primary decomposition of 0 in M. Then the map

$$\alpha : M \to \bigoplus_{j=1}^{t} M/M_j$$

$$m \to (m + M_1, \ldots, m + M_t)$$

is an embedding, which means that $A(M) \supseteq A(\bigoplus_{j=1}^{i} M/M_j)$

and hence

$$A(M) \subset \bigcup_{j=1}^{t} A(M/M_j), \text{ by Lemma 2.6.}$$

To see the opposite inclusion consider the homomorphism

$$\beta : \bigcap_{j \neq i} M_j \to M/M_i$$

$$m \to m + M_i$$

Now $\ker \beta = \{m \mid m \in \bigcup_{j \neq i}^{i} M_j\} = (0)$.

Thus β is an embedding, and so $A(\bigcap_{j \neq i} M_j) \subseteq A(M/M_i)$. Again by Lemma 2.2 $A(\bigcap_{j \neq i} M_j) \neq \phi$, And since $A(M/M_i)$ is singleton we then get

$A(\bigcap_{j \neq i} M_j) = A(M/M_i)$ for every i. Hence it follows that $\bigcup_{j=1}^{t} A(M/M_j) = \bigcup_{i=1}^{t} A(\bigcap_{j \neq t} M_j)$ and since $A(\bigcap_{j \neq i} M_j) \subset A(M)$ for each i, we finally get, $\bigcup_{j=1}^{t} A(M/M_j) \subseteq A(M)$. From Theorem 3.2, it follows that is R is a semiprime Goldie ring then the intersection of all the minimal prime ideals is zero.

Theorem 3.6—Let the module M be as in Lemma 2.8 and all its quotient modules are nonsingular over the finite dimensional ring R. Then given any submodule N of M, any $n \in Z^+$, n there exists a submodule M' of M such that

$$B^n \subseteq \sqrt{l(M/M')} \text{ and } B^n N = M' \cap N.$$

Proof : For any submodule N of M, any ideal B of R, an integer $n \in Z^+$, we get, by Lemma 2.8, a submodule M' of M such that

$$B^n N = M' \cap N \text{ and } A(N/B^n N) = A(M/M').$$

Also by Theorem 3.3 we have

$$\bigcap_{P \in A(M/M)} P = \sqrt{l(M/M')} \qquad \text{...(i)}$$

We claim that $B^n \subseteq l(N/B^n N)$. For, if $x_1, \ldots, x_n \in B$, then $x_1, \ldots, x_n \in B^n$. And therefore $x_1, \ldots, x_n N \subseteq B^n N$. Thus we have

$$x_1 \ldots x_n \in l(N/B^n N).$$

Therefore

$$B^n \subseteq l(N/B^n N) \qquad \text{...(ii)}$$

Again if $x \in l(N/B^n N)$,then $xN \subseteq B^n N$.

And if $P = l(D/B^n N) \in A(N/B^n N)$,

then $xD \subseteq xN \subseteq B^n N$. Thus $x \in l(D/B^n N)$

i e., $x \in P$ for all $P \in A(N/B^n N)$.

$$\text{Thus} \quad x \in \bigcap_{P \in A(M/M)} P = \sqrt{l(M/M')},$$

Hence $l(N/B^n N) \subseteq \sqrt{l(M/M')}$. Therefore we get $B^n \subseteq \sqrt{l(M/M')}$.

This theorem ultimately leads us to the Goldie analogue of Artin-Rees Theorem as stated.

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REGULAR RINGS

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Some equivalent conditions are established for the lattice of saturated sets of a commutative ring with identity, to be an infinitely meet distributive lattice. Using these, Noetherian regular rings are characterized.

1. INTRODUCTION

It is well known that in a commutative ring R with identity, the set $S(R)$ of all saturated subsets of R form a complete lattice. In this paper, we prove that R is Noetherian regular if and only if R is semiprime and $S(R)$ is a Boolean algebra. Further we establish some equivalent conditions in R for $S(R)$ to be an infinitely meet distributive lattice. Using these, we prove that R is Noetherian regular if and only if R is semiprime in which, every non-unit is a zero divisor and every completely irreducible saturated set is completely prime.

2. PRELIMINARIES

Throughout this paper, R denotes a commutative ring with identity. For every $a \in R$, the principal ideal generated by a is denoted by (a). An element $a \in R$ is said to be a semiprimary element if $\sqrt{(a)} = \text{rad}\ (a) = \{x \in R \mid x^n \in (a)$ for some $n \in Z^+\}$ is a proper prime ideal. A non empty subset F of R is said to be saturated if, for any $x, y \in R$, $xy \in F$ if and only if $x, y \in F$. The set of all saturated sets of R is denoted by $S(R)$. It should be mentioned that F is a saturated set in R if and only if its complement is a union of prime ideals. Obviously $S(R)$ is closed under arbitrary set intersection. For any $a \in R$, the smallest saturated set (Or principal saturated set generated by 'a') containing 'a' is denoted by $[a)$. It can be easily seen that $[a) = \{x \in R \mid xy = a^n$ for some $y \in R$ and $n \in Z^+ \cup \{0\}\}$. For any $\{F_\alpha\} \subseteq S(R)$, define $\bigvee_\alpha F_\alpha = \{x \in R \mid xy = f_{\alpha_1} \cdots f_{\alpha_n}$ for some $y \in R$ and $f_{\alpha_i} \in F_{\alpha_i}$; $i = 1, 2, \ldots, n\}$. Then $\bigvee F_\alpha \in S(R)$ and is the least upper bound of $\{F_\alpha\}$. Thus $S(R) = (S(R); \vee, \cap, [0), [1))$ is a complete lattice with $[0)$ as the greatest element and $[1)$ as the least element. Also it can be easily verified that if R is Noetherian regular, then S (R) is a Boolean algebra (see Theorem 2).

An element $F \in S(R)$ is said to be completely irreducible if $F = \bigcap_i F_i$ $(F_i \in S(R))$, then $F = F_i$ for some i. F is called prime if, for any $F_1, F_2 \in S(R)$,

$F_1 \cap F_2 \subseteq F$ implies either $F_i \subseteq F$ or $F_2 \subseteq F$. An element $F \in S(R)$ is called completely prime if $\bigcap_i F_i \subseteq F$ ($\{F_i\} \subseteq S(R)$) then $F_i \subseteq F$ for some i. F is said to be completely join irreducible if $F = \vee_i F_i$, then $F = F_i$ for some i. All ideals are assumed to be proper.

An element 'a' of a lattice $L = (L; \vee, \wedge)$ with 0 and 1 is said to be complemented if there exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$ (b is called a complement of a). L is called dual semicomplemented if for any non zero element $a \in L$, there is $b \in L$ such that $b \neq 1$ and $a \vee b = 1$. A complete lattice L is said to be infinitely meet distributive if for any $a \in L$ and $\{a_i\} \subseteq L$, $a \vee (\bigwedge_i a_i) = \bigwedge_i (a \vee a_i)$.

For all undefined terms in Ring theory and Lattice theory, the reader may refer to Atiyah and Macdonald[1], Birkhoff[3] and Burton[4].

4. Noetherian Regular Rings

We shall begin with the following lemma.

Lemma 1—Let $a, b \in R$. Then (i) $[b) \subseteq [a)$ if and only if every non-unit element x in $[b)$ is an element of $[a)$ and (ii) $\sqrt{(a)} \subseteq \sqrt{(b)}$ if and only if $[b) \subseteq [a)$.

Proof: The proof of (i) is obvious. Suppose $\sqrt{(a)} \subseteq \sqrt{(b)}$. Let $x \in [b)$ and be a non-unit. Then $xx_1 = b^n$ for some $x_1 \in R$ and $n \in Z^+$, so that $b \in \sqrt{(x)}$. As $a \in \sqrt{(a)} \subseteq \sqrt{(b)} \subseteq \sqrt{(x)}$, it follows that $a^m = xx_2$ for some $x_2 \in R$ and $m \in Z^+$; therefore $x \in [a)$ and hence $[b) \subseteq [a)$. Conversely, assume that $[b) \subseteq [a)$. If b is uuit, then obviously $\sqrt{(a)} \subseteq \sqrt{(b)}$. So assume that b is a non-unit element. Let $x \in \sqrt{(a)}$. Then $x^k = aa_1$ for some $k \in Z^+$ and $a_1 \in R$. So that $a \in [x)$ and hence $b \in [b) \subseteq [a) \subseteq [x)$. Therefore $bb_2 = x^i$ for some $i \in Z^+$. This shows that $x \in \sqrt{(b)}$ and thus $\sqrt{(a)} \subseteq \sqrt{(b)}$. This completes the proof of the lemma.

Lemma 2—Let $a \in R$ be a non-unit element. Then a is a semiprimary element if and only if, for any $x, y \in R$, $[a) \subseteq [x) \vee [y)$ implies either $[a) \subseteq [x)$ or $[a) \subseteq [y)$.

Proof: Observe that for any $x, y \in R$, $[x) \vee [y) = [xy)$. Now the result follows from lemma 1.

Lemma 3—Let $a, b \in R$. If $ab = 0$, then $[a) \cap [b) = [a + b)$.

Proof: Suppose $ab = 0$. Since $a(a + b) = a^2$, we get $a + b \in [a)$ and so $[a + b) \subseteq [a)$. Similarly $[a + b) \subseteq [b)$ and therefore $[a + b) \subseteq [a) \cap [b)$. Suppose x is a non-unit element of $[a) \cap [b)$. Then $xx_1 = a^n$ and $xx_2 = b^m$ for some $x_1, x_2 \in R$ and $n, m \in Z^+$. Since x is a divisor of a^n and b^m (we may assume $n = m$) it follows

that x is a divisor of $a^n + b^n = (a+b)^n$ and hence $x \in [a+b)$. This shows that $[a) \cap [b) = [a+b)$.

Lemma 4—Let F and $J \in S(R)$. Suppose $F \vee J = [0)$ and $F \cap J = [1)$. Then $F = [e)$ and $J = [1-e)$ for some idempotents $e, 1-e \in R$.

Proof: As $F \vee J = [0)$, we have $ab = 0$ for some $a \in F$ and $b \in J$. Also $[a) \cap [b) \subseteq F \cap J = [1) \subseteq [a) \cap [b)$, so $[a) \cap [b) = [a+b) = [1)$ by Lemma 3. Therefore $(a+b)x = 1$ for some $x \in R$. Let $e = ax$ and $f = bx$. Then $ef = 0$ and $e + f = 1$. Observe that e and f are idempotents. Since x is a unit, $x \in F$; so $e = ax \in F$. Similarly $f \in J$. We claim that $F = [e)$. Clearly $[e) \subseteq F$. Let $y \in F$. Then $ey \in F$ and hence $[ey + f) = [ey) \cap [f)$ (as $eyf = 0) \subseteq F \cap J = [1)$. Consequently $(ey + f)x_1 = 1$ for some $x_1 \in R$. Now $e = (ey+f)x_1 e = eyx_1 = y(ex_1)$, hence $y \in [e)$. This shows that $F = [e)$. Similarly $J = [f)$.

Lemma 5—Let L be a complete and infinitely meet distributive lattice. If L is dual semicomplemented, then L is a Boolean algebra.

Proof: Suppose L is dual semicomplemented. Since L is distributive, it is enough if we show that every element $a \in L$ has a complement in L. Let $a \in L$. Put $X = \{b \in L \mid a \vee b = 1\}$. As L is complete $\bigwedge_{b \in X} b$ exists. Let $a^* = \bigwedge_{b \in X} b$. Then $a \vee a^* = a \vee (\bigwedge_{b \in X} b) = \bigwedge_{b \in x} (a \vee b) = 1$. Now we claim that $a \wedge a^* = 0$. Suppose $a \wedge a^* \neq 0$. As L is dual semicomplemented, there exists $b \in L$ such that $b \neq 1$ and $(a \wedge a^*) \vee b = 1$. Clearly $a \vee b = 1$ and $a^* \vee b = 1$. Since $a \vee b = 1$, it follows that $b \in X$, so that $a^* \leqslant b$ and hence $b = 1$, a contradiction. Therefore $a \wedge a^* = 0$ and so a^* is a complement of a. This completes the proof of the lemma.

Lemma 6—Let $F \in S(R)$. Then F is the intersection of all completely irreducible saturated sets that contain it.

Proof: Similar to Theorem 2 of Ramana Murty[5].

Lemma 7—Let $S(R)$ be a distributive lattice and $[a)$ $(a \in R)$ be a joint irreducible element of $S(R)$. Then a is a semiprimary element.

Proof: Let $x, y \in R$. Suppose $[a) \subseteq [x) \vee [y)$. Then $[a) = (a) \cap ([x) \vee [y)) = ([a) \cap [x)) \vee [(a) \cap [y))$ (as $S(R)$ is distributive), so that either $[a) = [a] \cap [x)$ or $[a) = [a) \cap [y)$ and hence either $[a) \subseteq [x)$ or $[a) \subseteq [y)$. Consequently by Lemma 2, a is a semiprimary element.

Now we characterize regular rings and Noetherian regular rings as follows:

Theorem 1—R is regular if and only if R is semiprime and for each $a \in R$, $[a)$ has a complement in the lattice $S(R)$.

Proof: Suppose R is regular. Clearly R is semiprime. Let $a \in R$. As R is regular, $(a) = (e)$ for some idempotent $e \in R$. By Lemma 1, $[a) = [e)$. Obviously $[1-e)$ isa complement of $[a)$.

Conversely, assume that R is semiprime and for each $a \in R$, $[a)$ has a complement in $S(R)$. Let $a \in R$. Then by hypothesis and Lemma 4, $[a) = [e)$ for some idempotent $e \in R$. Now by Lemma 1, $\sqrt{(a)} = \sqrt{(e)} = (e)$. Since $e \in \sqrt{(a)}$, we have $e = e^n \in (a)$, so that $(a) = (e)$ and hence R is regular.

Remark 1 : The following examples shows that the conditions, assumed in Theorem 1, are independent.

Example 1—Let Z_4 = the ring of integers modulo 4. Then $S(Z_4) = \{[0), [1)\}$. It can be easily seen that for each $a \in Z_4$, $[a)$ is either $[0)$ or $[1)$ and so has a complement in $S(Z_4)$. Infact $S(Z_4)$ is a two element Boolean algebra. But Z_4 is not semiprime as $2^{-2} = 0$ and $\bar{2} \neq 0$.

Example 2—Let Z be the ring of integers. Clearly Z is semiprime. Also it can be easily seen that $[2)$ has no complement in $S(Z)$.

Theorem 2—R is Noetherian regular if and only if R is semiprime and $S(R)$ is a Boolean algebra.

PROOF : Suppose R is Noetherian regular. Clearly R is semiprime. As R is a Noetherian regular ring, it follows that R is a direct sum of fields, say $R = F_1 \oplus \dots \oplus F_n$. Since the prime ideals are $P_j = \underset{i \neq j}{\pi} F_i$, $j = 1, 2, \dots, n$, $S(R)$ is isomorphic to the Boolean algebra of subsets of $X = \{1, 2, \dots, n\}$.

Conversely, assume that R is semiprime and $S(R)$ is a Boolean algebra. By Theorem 1, R is regular. Let P be a prime ideal of R. Observe that $R - P$ is a saturated set. As $S(R)$ is a Boolean algebra, by Lemma 4, $R - P = [e)$ for some idempotent $e \in R$. Now it can be easily verified that $P = (1 - e)$. Again since every prime ideal is principal, it follows that every ideal is principal and hence R is Noetherian. This completes the proof of the theorem.

Remark 2 : From above examples in Remark 1, it can be easily shown that the conditions "(i) R is semiprime and (ii) $S(R)$ is a Boolean algebra", of Theorem 2 are independent.

Theorem 3—The following statements are equivalent :

(i) For every $a \in R$, there exists $a_1 \in [a)$ such that $aa_1 = P_1, \dots, P_n$; P_i's are semiprimary elements.

(ii) $S(R)$ is an infinitely meet distributive lattice.

(iii) Every completely irreducible saturated set is completely prime.

PROOF : (i) $\Rightarrow$ (ii). Suppose (i) holds. Let $F \in S(R)$ and $\{F_j \mid j \in \Delta\}$ be a family of saturated sets of $S(R)$. Clearly $F \vee (\bigcap_j F_j) \subseteq \bigcap_j (F \vee F_j)$. Let $a \in \bigcap_j (F \vee F_j)$. Then $a \in F \vee F_j$ for all $j \in \Delta$, so that for each $j \in \Delta$, there exist $a_j \in R$

such that $aa_j = f_j g_j$ for some $f_j \in F$ and $g_j \in F_j$. By (i), there exists $a_1 \in [a)$ and $P_i \in R$ $(i = 1, 2, \ldots, n)$ such that $aa_1 = P_1 . P_2 \ldots P_n$ where P_i's are semiprimary elements. Since for each $i \in \{1, 2, \ldots, n\}$, $P_i \in [aa_1) = [a) \vee [a_1) = [a) \subseteq [a) \vee [a_j) = [f_j) \vee [g_j)$ for all $j \in \Delta$, and P_i is a semiprimary element, by Lemma 2, each $P_i \in [f_j)$ or $P_1 \in (g)$ for all $j \in \Delta$; so that for each $i \in \{1, 2, \ldots, n\}$ $P_i \in F$ or $P_i \in F_j$ for all $j \in \Delta$ and hence $P_i \in F$ or $P_i \in \bigcap_j F_j$. Therefore each $P_i \in F \vee (\bigcap_j F_j)$. Consequently $a \in F \vee (\bigcap_j F_j)$. This shows that $F \vee (\bigcap_j F_j) = \bigcap_j (F \vee F_j)$ and therefore (ii) holds.

(ii) $\Leftrightarrow$ (ii) is similar to the proof of Balachandran[2] (Theorem 2, (iii) $\Leftrightarrow$ (iv)).

(ii) $\Rightarrow$ (i). Suppose (ii) holds. First we show that, for any $b \in R$, $[b)$ is the join of completely join irreducible saturated sets. Since $S(R)$ is a complete, infinitely meet distributive lattice, by Lemma 3 of Balachandran[2], it is enough if we show that, for any $F \in S(R)$, if $F \subset [b)$, then there exists $F_1 \in S(R)$ such that $F \subseteq F_1 \subset [b)$ and there is no element $J \in S(R)$ such that $F_1 \subset J \subset [b)$. Suppose $F \subset [b)$. Let $\mathscr{S} = \{J \in S(R) / F \subseteq J \subset [b)\}$. Then by Zorn's lemma, $\mathscr{S}$ contains a maximal element say F_1. Obviously F_1 satisfies the above condition and hence $[b)$ is the join of completely join of completely join irreducible saturated sets.

Now let $a \in R$. By the above argument, $[a)$ is the join of completely join irreducible saturated sets. Since every completely join irreducible saturated set is a principal saturated set, it follows that $[a) = \bigvee_\alpha [a_\alpha)$ where, for each α, $[a_\alpha)$ is completely join irreducible. By Lemma 7, each a_α is a semiprimary element. Since $a \in [a) = \bigvee_\alpha [a_\alpha)$, there is $b \in R$ such that $ab = f_1 f_2 \cdot f_n$ for some $f_i \in [a_{\alpha_i})$. Also for $i \in \{1, 2, \ldots, n\}$, there exist $g_i \in R$ such that $f_i g_i = a_{\alpha_i}^{n_i}$. Now $a(bg_1 \ldots g_n) = (f_1 \cdots f_n)(g_1 \cdots g_n) = a_{\alpha_1}^{n_1} \ldots a_{\alpha_n}^{n_n}$. Put $a_1 = bg_1 \ldots g_n$ and $P_i = a_{\alpha_i}^{n_i}$. Then $a_1 \in [a)$ and also each P_i is a semiprimary element. Thus for each $a \in R$, there exists $a_1 \in [a)$ such that $aa_1 = P_1 \ldots P_n$, (P_i's are semiprimary elements) and hence (i) holds. This completes the proof of the theorem.

Theorem 4—R is Noetherian regular if and only if R satisfies the following three conditions.

(i) R is semiprime.

(ii) Every nonunit in R is a zero divisor

(iii) Every completely irreducible saturated set is completely prime.

PROOF: Suppose R is Noetherian regular. Obviously R satisfies the conditions (i) and (ii). Again, $S(R)$ is a finite Boolean algebra; so (iii) of Theorem 4 is satisfied.

Conversely, assume that R satisfies the conditions (i), (ii) and (iii) of Theorem 4.

First, we show that $S(R)$ is dual semicomplemented. Let $F \in S(R)$ and $F \neq [1)$. Then there exists $a \in F$ such that a is a nonunit, so that by (ii), $ab = 0$ for some nonzero element $b \in R$. Obviously $F \vee [b) = [0)$ and $[b) \neq [0)$ (by semiprime property). Therefore $S(R)$ is dual semicomplemented. Again by the condition (iii), Theorem 3, and Lemma 5, $S(R)$ is a Boolean algebra. Now by (i) and Theorem 2, R is a Noetherian regular ring. This completes the proof of the theorem.

Remark 3 : We now show that the conditions (i), (ii) and (iii) of Theorem 4, are independent.

(a) Let Z_4 be the ring of integers modulo 4. Obviously Z_4 satisfies (ii) and (iii). But the condition (i) is not satisfied in Z_4.

(b) Consider the ring of integers Z. Clearly Z is semiprime and satisfies the condition (i) of Theorem 3, so that by Theorem 3, Z satisfies the condition (iii) of Theorem 4. Also 2, which is a nonunit, is a nonzero divisor. Therefore the condition (ii) of Theorem 4 is not satisfied in Z.

(c) Let X be an infinite set. Then $P(X)$ (Power set) is an infinite Boolean ring and so it is semiprime and every nonunit in $P(X)$ is a zero divisor. Also it can be easily seen that the zero element 0 (namely the empty set) cannot be written as a finite product of semiprimary elements and therefore by Theorem 3, every completely irreducible saturated set need not be completely prime. This shows that the condition (iii) of Theorem 4, is independent.

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A NOTE ON BIHARI TYPE INEQUALITIES IN TWO INDEPENDENT VARIABLES

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A Bihari type nonlinear integral inequality in two independent variables is considered for non-negative real-valued functions, by relaxing the conditions of sub-additivity, submultiplicativity and such other stingent conditions on the nonlinear function, imposed by earlier workers, which severely limit the class of admissible nonlinear functions. We prefer to impose some conditions on the free term outside the integral sign and obtain pointwise estimates, which are applicable to a larger class of nonlinear inequalities.

1. INTRODUCTION

We consider a Bihari-type inequality of the form

$$\phi(x, y) \leqslant a(x, y) + \int_0^x \int_0^y c(s, t)\, W[\phi(s, t)]\, ds\, dt,\ x \geqslant 0,\ y \geqslant 0 \quad \ldots(1.1)$$

in two independent variables.

There are many papers dealing with (1.1) or its one dimensional anologue or even with more general inequalities in one or two dimensions[1-5,7,8,11-13]. However, in all the earlier work, particularly in two dimensions, one or more of the following stingent conditions are imposed on W.

(1) W is sub-additive.

(2) W is sub-multiplicative.

(3) W is convex.

(4) $\frac{1}{v} W[u] \leqslant W[u/v],\ u \geqslant 0,\ v > 0$

(5) There exists a function ϕ continuous on $[0, \infty)$ with

$$W[u + \alpha v] \leqslant W[u] + \phi(\alpha) W[v] \text{ for } \alpha > 0, u, v \geqslant 0.$$

Our aim is to relax such conditions on W, which severely limit the class of admissible non-linear functions W in (1.1). Other conditions assumed by earlier workers for the study of (1.1) are

(6) The functions ϕ, a and c are real-valued continuous and non-negative for $x \geqslant 0, y \geqslant 0$.

(7) $W[u]$ is a real-valued positive continuous non-decreasing, function for $u > 0$.

We retain these assumptions (6) and (7) and add an assumption on the function $a(x, y)$, not taken by earlier workers.

(8) The derivatives $a_x(x, y)$, $a_y(x, y)$ and $a_{xy}(x, y)$ of the function $a(x, y)$ exist, are continuous for $x \geqslant 0, y \geqslant 0$ and $a_x(x, y) \geqslant 0$, $a_y(x, y) \geqslant 0$ while $a_{xy}(x, y) \leqslant 0$ there.

In the next section we obtain point-wise estimates for ϕ satisfying (1.1) subject to conditions (6), (7), (8) above. The importance of such results in the study of the qualitative behaviour of the solutions of differential and integral equations including the existence via monotone methods[6], uniqueness and continuous dependence on initial conditions[12], and stability[9], is well illustrated by earlier workers and so we do not discuss such possible applications of our results here.

2. The Main Result

The main result of this paper is contained in the following

Theorem 2.1—Let (1.1) hold subject to conditions (6), (7) and (8) of Section 1. Then

$$\phi(x, y) \leqslant \Omega^{-1}[\Omega\{a(0, y)\} + \Omega\{a(x, 0)\}$$
$$- \Omega\{a(0, 0)\} + \int_0^x \int_0^y c(s, t)\, ds\, dt], \; x \geqslant 0, y \geqslant 0 \qquad \ldots(2.1)$$

where

$$\Omega(r) = \int_{r_0}^{r} \frac{ds}{W[s]}. \qquad \ldots(2.2)$$

Ω^{-1} is the inverse function of Ω, and it is assumed that the quantity in square bracket on the right side of (2.1) belongs to the domain of Ω^{-1}.

Proof : Define

$$u(x, y) = a(x, y) + \int_0^x \int_0^y c(s, t) W[\phi(s, t)]\, ds\, dt$$

so that

$$u(0, y) = a(0, y),\ u_x(x, 0) = a_x(x, 0),\ u_y(0, y) = a_y(0, y).$$

Then

$$\text{(a)}\quad \phi(x, y) \leqslant u(x, y) \qquad \ldots(2.3)$$

and

$$\text{(b)}\quad u_x(x, y) = a_x(x, y) + \int_0^y c(x, t) W[\phi(x, t)]\, dt.$$

Since $a_{xy}(x, y) \leqslant 0$, we shall have

$$\begin{aligned} u_{xy}(x, y) &\leqslant c(x, y) W[\phi(x, y)] \\ &\leqslant c(x, y) W[u(x, y)]. \end{aligned}$$

Thus

$$\frac{u_{xy}(x, y)}{W[u(x, y)]} \leqslant c(x, y)$$

so that, since u_x, u_y are non-negative, we have

$$\frac{W[u(x, y)]\, u_{xy}(x, y)}{W^2[u(x, y)]} \leqslant c(x, y) + \frac{u_x(x, y)\, u_y(x, y)\, W'[u(x, y)]}{W^2[u(x, y)]}.$$

Keep x fixed, set $y = t$ and integrate w.r.f. t from 0 to y to get

$$\frac{u_x(x, y)}{W[u(x, y)]} \leqslant \frac{a_x(x, 0)}{W[a(x, 0)]} + \int_0^y c(x, t)\, dt.$$

Using the definition of Ω in (2.2), we obtain

$$\Omega_x[u(x, y)] \leqslant \frac{a_x(x, 0)}{W[a(x, 0)]} + \int_0^y c(x, t)\, dt.$$

Keeping y fixed, set $x = s$ and integrate w.r.t. s from 0 to x to get

$$\begin{aligned} \Omega[u(x, y)] \leqslant \Omega[a(0, y)] &+ \Omega[a(x, 0)] - \Omega[a(0, 0)] \\ &+ \int_0^x \int_0^y c(s, t)\, ds\, dt. \end{aligned}$$

Since Ω is strictly increasing, so is Ω^{-1}. Therefore

$$u(x, y) \leqslant \Omega^{-1}\Big[\Omega[a(x, 0)] + \Omega[a(0, y)] - \Omega[a(0, 0)] + \int_0^x \int_0^y c(s, t)\, ds\, dt\Big].$$

Substituting this bound on $u(x, y)$ in (2.3), we get the desired estimate (2.1).

3. Discussion

There is a Large class of functions, satisfying the condition (8) of section 1. For example, if F, f, g are continuously differentiable non-negative functions on $(0, \infty)$ for which $F'(\xi), f'(\xi), g'(\xi)$, are $\geqslant 0$, while $F''(\xi) \leqslant 0$ for $\xi \in (0, \infty)$, then the composite function

$$a(x, y) = F[f(x) + g(y)] \qquad \ldots(3.1)$$

satisfies the conditions (8).

$$F(\xi) = K\xi^{\alpha},\ 0 \leqslant \alpha \leqslant 1,\ \xi \in (0, \infty)$$

$$F(\xi) = K \log(1 + \xi)$$

where $K > 0$ is a constant, are some examples of F, while the set of real-valued functions on $[0, \infty)$ is rich with non-negative non-decreasing differentiable functions $[0, \infty)$. Thus a function having the form (3.1) in general, and a constant function in particular, can be always found to majorise the given free term $a(x, y)$ on a closed and bounded sub-domain of the first quadrant.

The characteristic initial value problem for a hyperbolic differential equation[10]

$$\phi_{xy}(x, y) = c(x, y)\, W[\phi(x, y)] \qquad \ldots(3.2)$$

[Walter[10], p. 146-151] when converted to an integral equation, generates a free term of the form $[f(x) + g(y)]$, which is of the form (3.1) if f and g are positive and non-decreasing, or else the free term can be majorised easily by a function of the form (3.1).

Again the functions of the form $c(x, y)\, W[\phi(x, y)]$ appearing on the right side of (3.2) can be used as majorising functions in the study of the qualitative theory of nonlinear differential and integral equations[10]. Thus Theorem 2.1 has a wide range of applicability.

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ABSOLUTE SUMMABILITY FACTORS FOR INFINITE SERIES

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In this paper a theorem on $|\bar{N}, p_n|_k$ summability factors has been proved under weaker conditions than the theorem of Bor[2].

§ 1. *Definition*—Let $\Sigma\, a_n$ be a given infinite series with the sequence of partial sums (s_n). Let (p_n) be a sequence of positive real constants such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \text{ as } n \to \infty, (P_{-k} = p_{-k} = 0, k \geqslant 1). \quad \text{...(1.1)}$$

The sequence-to-sequence transformation :

$$u_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v\, s_v \quad \text{...(1.2)}$$

defines the sequence (u_n) of $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n). The series $\Sigma\, a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geqslant 1$, if (see Bor[1])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \,|\, u_n - u_{n-1} \,|^k < \infty. \quad \text{...(1.3)}$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), then $|\bar{N}, p_n|_k$ summability is the same as $|C, 1|_k$ (resp. $|\bar{N}, p_n|$) summability.

§ 2. Quite recently Bor[2] proved the following theorem.

Theorem A—Let $k \geqslant 1$ and let (X_n) be a positive non-decreasing sequence and there be sequences (β_n) and (λ_n) such that

$$|\Delta \lambda_n| \leqslant \beta_n \quad \text{...(2.1)}$$

$$\beta_n \to 0 \text{ as } n \to \infty \quad \text{...(2.2)}$$

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$$\sum_{n=1}^{\infty} n \mid \Delta \beta_n \mid X_n < \infty \qquad \ldots(2.3)$$

$$\mid \lambda_n \mid X_n = O(1) \qquad \ldots(2.4)$$

and

$$\sum_{v=1}^{n} \frac{\mid s_v \mid^k}{v} = O(X_n) \text{ as } n \to \infty. \qquad \ldots(2.5)$$

Suppose further, the sequence (p_n) is such that

$$P_n = O(np_n) \qquad \ldots(2.6)$$

$$P_n \Delta p_n = O(p_n P_{n+1}). \qquad \ldots(2.7)$$

Then the series $\sum_{n=1}^{\infty} \frac{P_n a_n \lambda_n}{np_n}$ is summable $\mid \bar{N}, p_n \mid_k$.

If we take $k = 1$ in the Theorem A, then we get a theorem of Mishra and Srivastava[4].

§ 3. In this paper we shall prove the following theorem.

Theorem—Let $k \geqslant 1$ and let (X_n) be a positive non-decreasing sequence and the sequences (β_n) and (λ_n) are such that conditions (2.1)—(2.7) of the Theorem A are satisfied with the condition (2.5) replaced by

$$\sum_{v=1}^{n} \frac{\mid t_v \mid^k}{v} = O(X_n) \text{ as } n \to \infty \qquad \ldots(3.1)$$

where (t_n) is the nth $(C, 1)$ mean of the sequence (na_n), then the series $\sum_{n=1}^{\infty} \frac{P_n a_n \lambda_n}{np_n}$ is summable $\mid \bar{N}, p_n \mid_k$.

It should be noted that the condition (2.5) implies the condition (3.1), but the converse need not be true. So we are weakening the hypotheses replacing (2.5) by (3.1).

§ 4. We need the following lemmas for the proof of the theorem.

Lemma 1[3]—Let (X_n) be a positive non-decreasing sequence and the sequences (β_n) and (λ_n) are such that conditions (2.1)—(2.3) of the Theorem A are satisfied. Then

$$n \beta_n X_n = O(1) \qquad \ldots(4.1)$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \qquad \ldots(4.2)$$

Lemma 2—If the conditions (2.6) and (2.7) satisfy, then $\Delta (P_n/p_n n^2) = O(1/n^2)$.

PROOF : We have that

$$\begin{aligned}
\Delta\left(\frac{P_n}{n^2 p_n}\right) &= \frac{1}{(n+1)^2} \Delta\left(\frac{P_n}{p_n}\right) + \frac{P_n}{p_n} \cdot \frac{2n+1}{n^2 (n+1)^2} \\
&= \frac{1}{(n+1)^2} \left(\frac{P_n}{p_n} - \frac{P_n}{p_{n+1}} - 1\right) + \frac{P_n}{p_n} \cdot \frac{2n+1}{n^2 (n+1)^2} \\
&= \frac{1}{(n+1)^2} \left\{ P_n \left(\frac{1}{p_n} - \frac{1}{p_{n+1}}\right) - 1 \right\} + \frac{P_n}{p_n} \cdot \frac{2n+1}{n^2 (n+1)^2} \\
&= \frac{1}{(n+1)^2} \left\{ \frac{P_n - \Delta p_n}{p_n p_{n+1}} - 1 \right\} + \frac{P_n}{p_n} \cdot \frac{2n+1}{n^2 (n+1)^2} \\
&= O(1/n^2), \text{ by (2 6) and (2.7).}
\end{aligned}$$

This completes the proof of the lemma.

Remark : It should be noted that from the hypotheses of the Theorem, λ_n is bounded and $\Delta\lambda_n = O(1/n)$. This can be shown like this. Since (X_n) is non-decreasing, $X_n \geqslant X_0$, which is a positive constant. Hence (2.4) implies that (λ_n) is bounded. It also follows from (4.1) that $\beta_n = O(1/n)$ and thus (by (2.1)) that $\Delta\lambda_n = O(1/n)$.

§ 5. *Proof of the Theorem*—Let T_n be the $(\bar{N}, p_n)$ mean of the series $\sum_{n=1}^{\infty} P_n a_n \lambda_n / np_n$. Then, by definition, we have

$$T_n = (1/P_n) \sum_{v=1}^{n} p_v \sum_{r=1}^{v} P_r a_r \lambda_r / r p_r = (1/P_n) \sum_{v=1}^{n} (P_n - P_{v-1}) P_v a_v \lambda_v / v p_v. \qquad \ldots(5.1)$$

Then for $n \geqslant 1$

$$T_n - T_{n-1} = p_n (P_n P_{n-1})^{-1} \sum_{v=1}^{n} P_{v-1} P_v a_v \lambda_v / v p_v$$

$$= p_n (P_n P_{n-1})^{-1} \sum_{v=1}^{n} P_{v-1} P_v a_v v \lambda_v / v^2 p_v.$$

Using Abel's transformation we get

$$T_n - T_{n-1} = p_n (P_n - P_{n-1})^{-1} \sum_{v=1}^{n-1} \Delta (P_{v-1} P_v \lambda_v / v^2 p_v) \sum_{r=1}^{v} r a_r + \lambda_n / n^2 \sum_{v=1}^{n} v a_v$$

$$= - p_n (P_n - P_{n-1})^{-1} \sum_{\nu=1}^{n-1} (P_\nu / p_\nu)(\nu + 1) t_\nu p_\nu \lambda_\nu / \nu^2$$

$$+ p_n (P_n - P_{n-1})^{-1} \sum_{\nu=1}^{n-1} P_\nu P_\nu \Delta \lambda_\nu (\nu + 1) t_\nu / \nu^2 p_\nu$$

$$+ p_n (P_n - P_{n-1})^{-1} \sum_{\nu=1}^{n-1} P_\nu \lambda_{\nu+1} (\nu + 1) t_\nu \Delta (P_\nu / \nu^2 p_\nu)$$

$$+ \lambda_n t_n (n + 1)/n^2$$

$$= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \text{ say.}$$

To prove the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \mid T_{n,r} \mid^k < \infty, \text{ for } r = 1, 2, 3, 4. \qquad \ldots(5.2)$$

Now, applying Hölder's inequality, we have that

$$\sum_{n=2}^{m+1} (P_n/p_n)^{k-1} \mid T_{n,1} \mid^k = \sum_{n=2}^{m+1} \frac{p_n}{P_n P^k_{n-1}} \left| \sum_{\nu=1}^{n-1} \frac{P_\nu}{p_\nu} p_\nu t_\nu \lambda_\nu \frac{\nu + 1}{\nu} . \frac{1}{\nu} \right|^k$$

$$= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P^k_{n-1}} \left\{ \sum_{\nu=1}^{n-1} \frac{P_\nu}{p_\nu} p_\nu \mid t_\nu \mid \mid \lambda_\nu \mid \frac{1}{\nu} \right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \left\{ \sum_{\nu=1}^{n-1} \left(\frac{P_\nu}{p_\nu} \right)^k |\lambda_\nu|^k p_\nu |t_\nu|^k \frac{1}{\nu^k} \right\}$$

$$\times \left\{ \frac{1}{P_{n-1}} \sum_{\nu=1}^{n-1} p_\nu \right\}^{k-1}$$

$$= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \left\{ \sum_{\nu=1}^{n-1} \left(\frac{P_\nu}{p_\nu} \right)^k |\lambda_\nu|^k p_\nu |t_\nu|^k \frac{1}{\nu^k} \right\}$$

$$= O(1) \sum_{\nu=1}^{m} \left(\frac{P_\nu}{p_\nu} \right)^k \mid \lambda_\nu \mid^k p_\nu \mid t_\nu \mid^k \sum_{n=\nu+1}^{m+1} \frac{p_n}{P_n P_{n-1}}$$

$$= O(1) \sum_{\nu=1}^{m} \left(\frac{P_\nu}{p_\nu} \right)^k \mid \lambda_\nu \mid^{k-1} \mid \lambda_\nu \mid p_\nu \mid t_\nu \mid^k \frac{1}{P_\nu} \frac{1}{\nu^k}$$

$$= O(1)\sum_{\nu=1}^{m}\left(\frac{P_\nu}{p_\nu}\right)^{k-1} | \lambda_\nu || t_\nu |^k \frac{1}{\nu^k}$$

by $| \lambda_\nu | = O(1)$. On the other hand since $P_\nu = O(\nu p_\nu)$, by (2.6), we have that

$$\sum_{n=2}^{m+1} (P_n/p_n)^{k-1} | T_{n,1} |^k = O(1) \sum_{\nu=1}^{m} \nu^{k-1} \frac{1}{\nu^k} | \lambda_\nu || t_\nu |^k$$

$$= O(1) \sum_{\nu=1}^{m} | \lambda_\nu | \frac{| t_\nu |^k}{\nu}$$

$$= O(1) \sum_{\nu=1}^{m-1} | \Delta | \lambda_\nu | \sum_{r=1}^{\nu} r^{-1} | t_r |^k$$

$$+ O(1) | \lambda_m | \sum_{\nu=1}^{m} \nu^{-1} | t_\nu |^k$$

$$= O(1) \sum_{\nu=1}^{m-1} | \Delta\lambda_\nu | X_\nu + O(1) | \lambda_m | X_m$$

$$= O(1) \sum_{\nu=1}^{m-1} \beta_\nu X_\nu + O(1) | \lambda_m | X_m = O(1)$$

as $m \to \infty$, by (2.1), (2.4), (3.1) and (4.2).

Now, using the fact that $(P_\nu/\nu) = O(p_\nu)$, by (2.6), we have that

$$\sum_{n=2}^{m+1} (P_n/p_n)^{k-1} | T_{n,2} |^k = \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left| \sum_{\nu=1}^{n-1} \frac{P_\nu}{p_\nu} \Delta\lambda_\nu \frac{P_\nu}{\nu} t_\nu \frac{\nu+1}{\nu} \right|^k$$

$$= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left\{ \sum_{\nu=1}^{n-1} \frac{P_\nu}{p_\nu} | \Delta\lambda_\nu | p_\nu | t_\nu | \right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \left\{ \sum_{\nu=1}^{n-1} (P_\nu/p_\nu)^k | \Delta\lambda_\nu |^k | t_\nu |^k p_\nu \right\}$$

$$\times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1}$$

$$= O(1)\sum_{v=1}^{m} (P_v/p_v)^k \mid \Delta\lambda_v \mid^k \mid t_v \mid^k p_v \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} (P_v/p_v)^{k-1} \mid \Delta\lambda_v \mid^k \mid t_v \mid^k$$

$$= O(1) \sum_{v=1}^{m} (P_v/p_v)^{k-1} \mid \Delta\lambda_v \mid^{k-1} \mid \Delta\lambda_v \mid \mid t_v \mid^k$$

$$= O(1)\sum_{v=1}^{m} v^{k-1} \frac{1}{v^{k-1}} \mid \Delta\lambda_v \mid\mid t_v \mid^k$$

$$= O(1) \sum_{v=1}^{m} \beta_v \mid t_v \mid^k$$

by (2.1), (2.6) and $\Delta\lambda_v = O(1/v)$.

Hence

$$\sum_{n=2}^{m+1} (P_n/p_n)^{k-1} \mid T_{n,2} \mid^k = O(1) \sum_{v=1}^{m} \beta_v \mid t_v \mid^k = O(1) \sum_{v=1}^{m} v\, \beta_v \frac{\mid t_v \mid^k}{v}$$

$$= O(1)\sum_{v=1}^{m-1} \Delta(v\, \beta_v) \sum_{r=1}^{v} \frac{\mid t_r \mid^k}{r}$$

$$+ O(1)\, m\, \beta_v \sum_{v=1}^{m} \frac{\mid t_v \mid^k}{v}$$

$$= O(1) \sum_{v=1}^{m-1} v \mid \Delta\beta_v \mid X_v + O(1) \sum_{v=1}^{m-1} \beta_v X_v$$

$$+ O(1)\, m\, \beta_m X_m = O(1)$$

by (2.3), (3.1), (4.1) and (4.2).

Now, since $\Delta(P^2/p_v\, v_v) = O(1/v^2)$, by lemma 2, we have that

$$\sum_{n=2}^{m+1} (P_n/p_n)^{k-1} \mid T_{n,3} \mid^k = \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left|\sum_{v=1}^{n-1} P_v\, \Delta(P_v/p_v\, v^2)\, \lambda_{v+1}\, (v+1)\, t_v\right|^k$$

$$= O(1)\sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k}\left\{\sum_{v=1}^{n-1} P_v \mid \lambda_{v+1} \mid\mid t_v \mid \frac{1}{v} \frac{v+1}{v}\right\}^k$$

$$= O(1)\sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k}\left\{\sum_{v=1}^{n-1} \frac{P_v}{p_v}\, p_v \mid \lambda_{v+1} \mid \frac{1}{v} \mid t_v \mid\right\}^k$$

$$= O(1)\sum_{n=1}^{m+1} \frac{p_n}{P_n P_{n-1}} \left\{\sum_{v=1}^{n-1} (P_v/p_v)^k \frac{1}{v^k} \mid \lambda_{v+1}\mid^k \mid t_v \mid^k\right\}$$

$$\times \left\{\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v\right\}^{k-1}$$

$$= O(1)\sum_{v=1}^{m} (P_v/p_v)^k\, p_v\, \frac{1}{v^k} \mid \lambda_{v+1} \mid^{k-1} \mid \lambda_{v+1} \mid\mid t_v \mid^k$$

$$\sum_{n=v+1}^{m+1} \frac{p_n}{P_v\, P_{n-1}} = O(1) \sum_{v=1}^{m} (P_v/p_v)^{k-1} \frac{1}{v^k} \mid \lambda_{v+1} \mid\mid t_v \mid^k$$

$$= O(1) \sum_{v=1}^{m} v^{k-1} \frac{1}{v^k} \mid \lambda_{v+1} \mid \mid t_v \mid^k$$

$$= O(1)\sum_{v=1}^{m} \mid \lambda_{v+1} \mid \frac{\mid t_v \mid^k}{v} = O(1) \sum_{v=1}^{m-1} \Delta \mid \lambda_{v+1} \mid$$

$$\sum_{r=1}^{v} \frac{\mid t_r \mid^k}{r} + O(1) \mid \lambda_{m+1} \mid \sum_{v=1}^{m} \frac{\mid t_v \mid^k}{v}$$

$$= O(1) \sum_{v=1}^{m-1} \mid \Delta\lambda_{v+1} \mid X_v + O(1) \mid \lambda_{m+1} \mid X_m.$$

Since (X_v) is non-decreasing we have that

$$\sum_{n=2}^{m+1} (P_n/p_n)^{k-1} \mid T_{n,3} \mid^k = O(1) \sum_{v=1}^{m-1} \mid \Delta\lambda_{v+1} \mid X_{v+1} + O(1) \mid \lambda_{m+1} \mid X_{m+1}$$

$$= O(1) \sum_{v=2}^{m} \mid \Delta\lambda_v \mid X_v + O(1) \mid \lambda_{m+1} \mid X_{m+1}$$

$$= O(1) \sum_{v=1}^{m} \beta_v X_v + O(1) \mid \lambda_{m+1} \mid X_{m+1}$$

$= O(1)$ as $m \to \infty$, by (2.1), (2.4), (2.6), (3.1) and (4.2).

Finally, we have that

$$\sum_{n=1}^{m} (P_n/p_n)^{k-1} \mid T_{n,4} \mid^k = \sum_{n=1}^{m} (P_n/p_n)^{k-1} \left(\frac{n+1}{n}\right)^k \frac{1}{n^k} \mid \lambda_n \mid^k \mid t_n \mid^k$$

$$= O(1) \sum_{n=1}^{m} n^{k-1} \frac{1}{n^k} \mid \lambda_n \mid^{k-1} \mid \lambda_n \| t_n \mid^k$$

$$= O(1) \sum_{n=1}^{m} \mid \lambda_n \mid \frac{\mid t_n \mid^k}{n}.$$

As in $T_{n,3}$, we have that

$$\sum_{n=1}^{m} (P_n/p_n)^{k-1} \mid T_{n,4} \mid^k = O(1) \text{ as } m \to \infty.$$

Therefore, we get that

$$\sum_{n=1}^{m} (P_n/p_n)^{k-1} \mid T_{n,r} \mid^k = O(1) \text{ as } m \to \infty, \text{ for } r = 1, 2, 3, 4.$$

This completes the proof of theorem.

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ON THE INTEGRALS OF BMOA FUNCTIONS

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In the present paper we prove that the integrals of BMOA functions form, under the usual multiplication and with a suitable norm, a Banach algebra.

§1. BMOA functions (analytic functions of bounded mean oscillation) are the functions $f \in H^2$ with

$$\sup_{\zeta \in D} \|f_\zeta\|_2 < \infty, \text{ where}$$

$$f_\zeta(z) = f\left(\frac{z+\zeta}{1+\bar{\zeta}z}\right) - f(\zeta) \text{ for } \zeta, z \text{ in } D = \{z \in \mathbb{C}, |z| < 1\}. \quad ...(1)$$

With $\|f\|_* = |f(0)| + \sup_{\zeta \in D} \|f_\zeta\|_2$ the BMOA functions form a Banach space, topologically equivalent to $(H^1)^*$ (Fefferman and Stein[4]).

Further we let $\mathscr{B}$ denote the Banach space of Bloch functions, i. e. the functions analytic in D, with

$$\|f\|_{\mathscr{B}} = |f(0)| + \sup_{z \in D} (1 - |z|^2) |f'(z)| < \infty \quad ...(2)$$

and recall the well-known inclusion BMOA $\subset \mathscr{B}$ (Pommerenke[5], p. 592).

The space BMOA is not an algebra under the usual multiplication. For instance the square of the BMOA function $\log(1-z)$ is not even a Bloch function, because it violates (2). On the other hand Anderson *et al.*[1] (p. 29), have shown, that the integrals of Bloch functions have this algebra property. It is also known that the Bloch functions coincide with the analytic functions in D, which are BMO as functions of two real variables[2].

So it is natural to ask whether the integrals of BMOA functions form an algebra. This is exactly the question answered in the present note.

Using as auxiliary Lemma, which is given separately in section 2, we prove in section 3 that the class of integrals of BMOA functions is, under the usual multiplication and with a suitable norm, a Banach algebra.

§ 2. Let $f \in H^2$. For the functions f_ζ defined in (1) we have after a short calculation

$$\| f_\zeta \|_2^2 = \frac{2}{n} \iint_D | f'(z) |^2 \log \left| \frac{1 - \bar{\zeta} z}{\zeta - z} \right| d_z \Omega \qquad \ldots(3)$$

and consequently,

$$f \in \text{BMOA} \Leftrightarrow \| f \|_* = | f(0) | + \sup_{\zeta \in D} \left\{ \frac{2}{n} \iint_D | f'(z) |^2 \log \right.$$

$$\left. \times \left| \frac{1 - \bar{\zeta} z}{\zeta - z} \right| d_z \Omega \right\}^{1/2} < \infty \qquad \ldots(4)$$

where $d_z \Omega = dx\, dy$, if $z = x + iy$.

Since it is not easy to handle the logarithm under the integral sign in (3), we shall now give an equivalent expression for $\| f_\zeta \|_2^2$, which does not involve any logarithm, while the derivative of f is still present.

Lemma—If $f \in H^2$, then for all $\zeta \in D$

$$m_1 \| f_\zeta \|_2^2 \leqslant \iint_D | f'(z) |^2 \frac{(1 - | \zeta |^2)(1 - | z |^2)}{| 1 - \bar{\zeta} z |^2} d_z \Omega \leqslant m_2 \| f_\zeta \|_2^2$$

where m_1, m_2 are absolute constants.

Proof : By taking into account (3) and using the transformation $w = \dfrac{z - \zeta}{1 - \bar{\zeta} z}$,

we have to prove following estimate for all $f \in H^2$ and $\zeta \in D$:

$$m_1 \iint_D | f'_\zeta (z) |^2 \log \frac{1}{| z |} d_z \Omega \leqslant \iint_D | f'_\zeta (z) |^2 (1 - | z |^2) \, d_z \Omega$$

$$\leqslant m_2 \iint_D | f'_\zeta (z) |^2 \log \frac{1}{| z |} d_z \Omega. \qquad \ldots(5)$$

The right-hand inequality is obvious, because

$$\tfrac{1}{2} (1 - | z |^2) < 1 - | z | < \log \frac{1}{| z |} \text{ for every } z \in D.$$

For the left-hand inequality we note that if $\frac{1}{2} < | z | < 1$,

then

$$\log \frac{1}{|z|} < \frac{1}{|z|} - 1 < 2(1 - |z|^2).$$

Hence for all $\zeta \in D$

$$\frac{1}{2} \iint_{\frac{1}{2} < |z| < 1} |f_\zeta'(z)|^2 \log \frac{1}{|z|} d_z \Omega \leqslant \iint_D |f_\zeta'(z)|^2 (1 - |z|^2) d_z \Omega.$$

For any $\zeta \in D$ we set

$$L = \iint_D |f_\zeta'(z)|^2 (1 - |z|^2) d_z \Omega = \int_0^1 \varphi(\rho)(1 - \rho^2) d\rho,$$

where

$$z = \rho e^{i\vartheta} \text{ and } \varphi(\rho) = \rho \int_0^{2\pi} |f_\zeta'(\rho e^{i\vartheta})|^2 d\vartheta.$$

A classical result of Hardy asserts that $\varphi(\rho)$ is monotonically increasing in $[0, 1)$ (Duren[3], p. 9). From this we obtain for an arbitrary $r \in [0, 1)$

$$L \geqslant \int_r^1 \varphi(\rho)(1 - \rho^2) d\rho \geqslant \varphi(r) \int_r^1 (1 - \rho^2) d\rho > \varphi(r) \tfrac{2}{3} (1 - r)^2, \text{ or}$$

$$\varphi(r) < \tfrac{3}{2} \frac{1}{(1 - r)^2} \leqslant 6L, \text{ when } 0 \leqslant r \leqslant \tfrac{1}{2}.$$

It follows that for every $\zeta \in D$

$$\iint_{0 \leqslant |z| < 1/2} |f_\zeta'(z)|^2 \log \frac{1}{|z|} d_z \Omega = \int_0^{1/2} \varphi(r) \log \frac{1}{r} dr < 6L \int_0^{1/2} \log \frac{1}{r} dr$$

$$< \text{const.} \iint_D |f_\zeta'(z)|^2 (1 - |z|^2) d_z \Omega$$

which completes the proof of our Lemma.

§3. We are now able to prove the main result of this note :

Theorem—Let $f_i \in$ BMOA, $i = 1, 2$ and

$$F_i(z) = \int_0^z f_i(\zeta) d\zeta, \; z \in D.$$

Then there is a constant K such that the norm $\|F_i\| = K\,\|f_i\|_*$ satisfies the inequality

$$\|F_1\,F_2\| \leqslant \|F_1\|\,\|F_2\|.$$

PROOF : Setting

$$F_1(z)\,F_2(z) = \int_0^z g(z)\,dz,\ z \in D,$$

we have to prove that

$$\|g\|_* \leqslant K\,\|f_1\|_*\,\|f_2\|_*.$$

The first step in the proof is the following simple estimate :

$$|g'(z)|^2 \leqslant (|f_1'(z)||F_2(z)| + 2|f_1(z)||f_2(z)| + |f_2'(z)||F_1 z)^2$$

$$\leqslant \text{const. } |f_1'(z)|^2\|F_2\|_\infty^2 + \text{const. } |f_2'(z)|^2\|F_1\|_\infty^2$$

$$+ \text{const. } |f_1(z)|^2\,|f_2(z)|^2,\ z \in D. \qquad \ldots(6)$$

For every $f \in$ BMOA it is known that

$$|f(z)| \leqslant \|f\|_* \log\frac{1}{1-|z|} + |f(0)|,\ z \in D. \qquad \ldots(7)$$

PROOF : $f \in$ BMOA implies f Bloch and hence $|f'(z)| \leqslant \|f\|_B \dfrac{1}{1-|z|^2} \leqslant \|f\|_*$ $\dfrac{1}{1-|z|}$ from which the above inequality follows easily by integration (see Pommerenke[5], p. 592, for the details)].

Using (7) and the fact that F_1, F_2 are analytic in D and continuous in $\bar{D}$ (Duren[8], p. 42) we obtain

$$\|F_i\|_\infty = |F_i(e^{i\vartheta_0})| \leqslant \int_0^1 |f_i(se^{i\vartheta_0})|\,ds \leqslant \text{const. } \|f_i\|_* \int_0^1 \log\frac{1}{1-s}\,ds$$

$$+ |f_i(0)| \leqslant \text{const. } \|f_i\|_*,\ i = 1, 2. \qquad \ldots(8)$$

Setting now for brevity

$$A(\zeta, z) = \frac{(1-|\zeta|^2)(1-|z|^2)}{|1-\bar{\zeta}z|^2},\ \zeta, z \in D.$$

$$I(\xi) = \iint_D |g'(z)|^2 A(\zeta, z)\,d_z\,\Omega,$$

we deduce from (6), (8) and our lemma

$$I(\zeta) \leqslant \text{const. } \|f_1\|_*^2\,\|f_2\|_*^2 + \text{const. } \iint_D |f_1|^2\,|f_2|^2 A(\zeta, z)\,d_z\,\Omega. \qquad \ldots(9)$$

On the other hand the obvious inequality $A\ (\zeta, z) < (1 + |\zeta|)(1 + |z|) < 4$ and (7) yield

$$\iint_D |f_1(z)|^2 |f_2(z)|^2 A(\zeta, z)\, d_z\Omega \leqslant \|f_1\|_*^2 \|f_2\|_*^2 \iint_D \left(\log \frac{1}{1-|z|} + 1\right)^2 d_z\Omega$$

$$\leqslant \text{const. } \|f_1\|_*^2 \ \|f_2\|_*^2 \qquad \ldots(10)$$

as an easy calulation shows.

Since $g(0) = 0$, (4), (9) (10) and again the Lemma imply

$$\|g\|_*^2 = \sup_{\zeta\in D} I(\zeta) \leqslant \text{ const } \|f_1\|_*^2 \ \|f_2\|_*^2,$$

and the proof is complete.

An immediate consequence of our theorem is the following corollary :

Corollary —The integrals of BMOA functions form under the natural multiplication and with the norm defined in the statement of the theorem a Banach algebra without unit element.

Of course we can supply this algebra with a unit element in the canonical fashion (Rudin[6], p. 228).

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A CLASS OF EXACT SOLUTIONS IN PLANE ROTATING MHD FLUID FLOWS

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A legendre transformation is employed to obtain a partial differential equation of second order which is exploited to obtain solutions for variably inclined plane rotating viscous incompressible flows with orthogonal magnetic and velocity fields. Lastly, radial, vortex and hyperbolic flows are discussed.

1. INTRODUCTION

Transformation methods has been extensively used in literature for analysing and classifying the differential equations. Mathematical complexity of the phenomenon induced by many researcher to adopt a rather useful technique of investigating special classes of flows. These special classes of flows yielded various soluble second order mathematical structures. In the area of fluid mechanics Martin[3], used a type of transformation of variables from physical plane to velocity plane, to study the plane flows. An excellent survey of the method with application to various other fields was given by Power and Walker[4], Bloomer *et al.*[5], Singh and Tripathi[4], determined the flows geometries when the velocity magnitude is constant along each stream lines.

In present work using legendre transformation, we obtain a linear partial differential equation of second order, a solution of which leads to the velocity field of a flow. This approach is illustrated by considering the different problems and solutions for radial, vortex and hyperbolic flows are obtained.

2. FLOW EQUATIONS

The steady flow in a rotating reference frame of a homogeneous, incompressible viscous fluid with infinite electrical conductivity is governed by the system of equations[2]

$$\text{div } \rho \mathbf{v} = 0 \qquad \ldots(2.1)$$

$$\rho\,[\mathbf{v}.\text{ grad})\,\mathbf{v} + 2\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})] = -\text{ grad } p + \eta\,\nabla^2\,\mathbf{v} + \mu\,\mathbf{J} \times \mathbf{H} \qquad \ldots(2.2)$$

$$\text{curl } (\mathbf{v} \times \mathbf{H}) = 0 \qquad \ldots(2.3)$$

and

$$\text{div } \mathbf{H} = 0 \qquad \text{...(2.4)}$$

where **v** denotes the velocity vector; **H** the magnetic field vector; p the fluid pressure; **r** the radius vector; **ω** angular viscosity vector; ρ, η, μ are fluid density, coefficient of viscosity and magnetic permeability respectively.

In the case of plane flows **H** is in the plane of flow and orthogonal to the velocity field. Introducing the functions

$$\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \; J = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}$$

$$h = \tfrac{1}{2} \rho v^2 + p' - \tfrac{1}{2} \mid \omega \times r \mid^2 \qquad \text{...(2.5)}$$

where p' is the reduced pressure, $p' = p - \frac{1}{2} \mid \omega \times \mathbf{r} \mid^2$ and the last term being the centrifugal contribution of the pressure[8] (p. 69). Let $\alpha = \alpha(x, y)$ be the variable angle such that $\alpha(x, y) \neq 0$ for every (x, y) in the region of flow, eqns. (2.3) yields

$$uH_2 - vH_1 = qH \sin \alpha = k \qquad \text{...(2.6)}$$

$$uH_1 + vH_2 = qH \cos \alpha = k \cot \alpha$$

where

$$H = (H_1^2 + H_2^2)^{1/2} \text{ and } q = (\mu^2 + v^2)^{1/2}.$$

Considering these as two linear algebraic equations in the unknown H_1, H_2, such that

$$H_1 = \frac{k}{q^2}(u \cot \alpha - v), \; H_2 = \frac{k}{q^2}(v \cot \alpha + u). \qquad \text{...(2.7)}$$

The above system of equations (2.1) — (2.4) is replaced by the following system :

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{...(2.8)}$$

$$\eta \frac{\partial \xi}{\partial y} - \rho \xi v - 2\rho\omega v + \frac{\mu k}{q^2}(v \cot \alpha + u) j = -\frac{\partial h}{\partial x} \qquad \text{...(2.9)}$$

$$\eta \frac{\partial \xi}{\partial x} - \rho \xi u - 2\rho\omega u - \frac{\mu k}{q^2}(u \cot \alpha - v) j = \frac{\partial h}{\partial y} \qquad \text{...(2.10)}$$

$$(v^2 - u^2 - 2uv \cot \alpha)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + (v^2 \cot \alpha - u^2 \cot \alpha + 2uv)$$

$$\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) + q^2\left(u \frac{\partial}{\partial x} \cot \alpha + v \frac{\partial}{\partial y} \cot \alpha\right) = 0 \qquad \text{...(2.11)}$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \xi \qquad \text{...(2.12)}$$

and

$$k \frac{\partial}{\partial x}\left(\frac{u \cot \alpha + u}{q^2}\right) - k \frac{\partial}{\partial y}\left(\frac{u \cot \alpha - v}{q^2}\right) = j. \quad \ldots(2.13)$$

This approach leads to us the study of flows after hodograph transformations in the hodograph plane.

3. Generalities

As mentioned in the flow equations $u = u(x, y)$, $v = v(x, y)$, then Jacobian

$$J(x, y) = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \neq 0; 0 < | \mathbf{J} | \infty. \quad \ldots(3.1)$$

We may consider x, y as function u, v by means of $x = x\ (u, v)$, $y = y\ (u, v)$, then we have

$$J(x, y) = \frac{\partial (u, v)}{\partial (x, y)} = \left[\frac{\partial (x, y)}{\partial (u, v)}\right]^{-1} = j(u, v). \quad \ldots(3.2)$$

Employing these transformation relations for the first order partial derivatives and eqn. (3.2) in the system of equations (2.8)—(2.13), the transformed system of partial differential equations in the (u, v) plane is,

$$\frac{\partial x}{\partial u} + \frac{\partial y}{\partial v} = 0 \quad \ldots(3.3)$$

$$\eta J \frac{\partial (x, \xi)}{\partial (u, v)} - \rho \xi v - 2\rho\omega v + \frac{\mu k^2}{q^2}(v \cot \alpha + u)\, j = -J \frac{\partial (h, y)}{\partial (u, v)} \quad \ldots(3.4)$$

$$\eta J \frac{\partial (\xi, y)}{\partial (u, v)} - \rho \xi u - 2\rho\omega u + \frac{\mu k^2}{q^2}(u \cot \alpha - v)\, j = J \frac{\partial (x, h)}{\partial (u, v)} \quad \ldots(3.5)$$

$$\begin{aligned}
&\left[v(u^2 + v^2)\frac{\partial}{\partial v}\cot \alpha - 2uv + (u^2 - v^2)\cot \alpha\right]\frac{\partial x}{\partial u} \\
&+ \left[2uv \cot \alpha + (u^2 - v^2) - v(u^2 + v^2)\frac{\partial}{\partial u}\cot \alpha\right]\frac{\partial x}{\partial v} \\
&+ \left[2uv \cot \alpha + (u^2 - v^2) - u(u^2 + v^2)\frac{\partial}{\partial v}\cot \alpha\right]\frac{\partial y}{\partial u} \\
&+ \left[u(u^2 + v^2)\frac{\partial}{\partial u}\cot \alpha + 2uv + (v^2 - u^2)\cot \alpha\right]\frac{\partial y}{\partial v} = 0
\end{aligned} \quad \ldots(3.6)$$

$$J\left(\frac{\partial x}{\partial v} - \frac{\partial y}{\partial u}\right) = \xi \quad \ldots(3.7)$$

$$kJ\left[\frac{\partial\,((v\cot\alpha+u)q^{-2},\,y)}{\partial\,(u,v)}\quad\frac{\partial\,(x,\,(u\cot\alpha-v)\,q^{-2})}{\partial\,(u,\,v)}\right]=J. \qquad \ldots(3.8)$$

The equation of continuity implies the existence of stream function $\psi\,(x,\,y)$ so that

$$d\psi=-\,vdx+udy$$

or

$$\frac{\partial\psi}{\partial x}=-\,v,\quad\frac{\partial\psi}{\partial y}=u. \qquad \ldots(3.9)$$

Likewise, equation (3.3) implies the existence of a function $L\,(u,\,v)$, called the Legendre transfer function of the stream function $\psi\,(x,\,y)$, so that

$$\frac{\partial L}{\partial u}=-\,y,\quad\frac{\partial L}{\partial v}=x \qquad \ldots(3.10)$$

and the two functions $\psi\,(x,\,y)$, $L\,(u,\,v)$ are related by

$$L\,(u,\,v)=vx-uy+\psi\,(x,\,y). \qquad \ldots(3.11)$$

Introducing $L\,(u,\,v)$ into the system eqns. (3.3) to (3.8) with J given by (3.2), it follows that the eqns. (3.3) is identically satisfied and this system may be replaced by,

$$J\,\frac{\partial\left(\dfrac{\partial L}{\partial v},\,\xi\right)}{\partial\,(u,\,v)}-\rho\,v-2\rho\omega v+\frac{\mu k}{q^2}\,(v\cot\alpha+u)\,j=J\,\frac{\partial\left(h,\,\dfrac{\partial L}{\partial u}\right)}{\partial\,(u,v)} \qquad \ldots(3.12)$$

$$J\,\frac{\partial\,(\xi,\,\dfrac{\partial L}{\partial u})}{\partial\,(u,\,v)}+\rho\,u+2\rho\omega u-\frac{\mu k}{q^2}\,(u\cot\alpha-v)\,J=\frac{\partial\left(\dfrac{\partial L}{\partial v},\,h\right)}{\partial\,(u,\,v)} \qquad \ldots(3.13)$$

$$\begin{aligned}&[v^2-u^2-2uv\cot\alpha-u\,(u^2+v^2)\,\frac{\partial}{\partial v}\,\cot\alpha]\,\frac{\partial^2 L}{\partial u^2}\\&+[2\,(u^2-v^2)\cot\alpha-uv-u\,(u^2+v^2)\,\frac{\partial}{\partial u}\cot\alpha+v\,(u^2+v^2)\\&\frac{\partial}{\partial v}\cot\alpha]\,\frac{\partial^2 L}{\partial u\,\partial v}+[u^2-v^2-2uv\,\cot\alpha-v\,(u^2+v^2)\\&\frac{\partial}{\partial u}\cot\alpha]\,\frac{\partial^2 L}{\partial v^2}=0.\end{aligned} \qquad \ldots(3.14)$$

$$J\left[\frac{\partial^2 L}{\partial u^2}+\frac{\partial^2 L}{\partial v^2}\right] \qquad \ldots(3.15)$$

$$=kJ\left[\frac{\left(\dfrac{\partial L}{\partial u},\,\dfrac{v\cot\alpha+u}{q^2}\right)}{\partial\,(u,\,v)}+\frac{\partial\left(\dfrac{\partial L}{\partial v},\,\dfrac{v-u\cot\alpha}{q^2}\right)}{\partial\,(u,\,v)}\right]J \qquad \ldots(3.16)$$

and

$$\left[\frac{\partial^2 L}{\partial u^2}\frac{\partial^2 L}{\partial v^2}-\left(\frac{\partial^2 L}{\partial u\,\partial v}\right)^2\right]^{-1}=J \qquad \ldots(3.17)$$

for the seven functions $L(u, v)$, $h(u, v)$, (u, v), $j(u, v)$, $\omega(u, v)$, $\alpha(u, v)$ and $J(u, v)$. We now define

$$Q_1(u, v) = \frac{\partial\left(\frac{\partial L}{\partial v}, \xi\right)}{\partial(u, v)} = \frac{\partial\left(\frac{\partial L}{\partial v}, J\frac{\partial^2 L}{\partial u^2}+J\frac{\partial^2 L}{\partial v^2}\right)}{\partial(u, v)} \qquad \ldots(3.18)$$

$$Q_2(u, v) = \frac{\left(\partial\frac{\partial L}{\partial u}, \xi\right)}{\partial(u, v)} = \frac{\partial\left(\frac{\partial L}{\partial u}, J\frac{\partial^2 L}{\partial u^2}+J\frac{\partial^2 L}{\partial v^2}\right)}{\partial(u, v)} \qquad \ldots(3.19)$$

and integrability condition yields

$$\frac{\partial^2 h}{\partial x\,\partial y} = \frac{\partial^2 h}{\partial y\,\partial x} \qquad \ldots(3.20)$$

to eliminate $h(u, v)$ from equations (3.12) and (3.13).

4. Vortex Flow

An appropriate choice, resembles the simplest solution of eqn. (3.14) as,

$$L(u, v) = M_1 q^2 + M_2,\ M_1 \neq 0$$

$$\alpha(u, v) = \cot^{-1}(M_3 q^2 + M_4),\ M_3 \neq 0 \qquad \ldots(4.1)$$

where M's are arbitrary constants.

This constraint together with eqn. (3.10) ensure that

$$x = \frac{\partial L}{\partial v} = 2M_1 v,\ y = -\frac{\partial L}{\partial u} = -2M_1 u \qquad \ldots(4.2)$$

and therefore the velocity field is given by,

$$u = -\frac{y}{2M_1},\ v = \frac{x}{2M_1}. \qquad \ldots(4.3)$$

This relation represent a circulatory flow. And the velocity of the fluid vary directly to its radial distance from the central axis. The variable angle between the velocity and magnetic fields in the physical plane, is obtained by using (4.3) in (4.1)

$$\alpha(x, y) = \cot^{-1}\left[\frac{M_4}{4M_1^2} r^2 + M_4\right] \text{with } x^2 + y^2 = r^2. \qquad \ldots(4.4)$$

Using (4.3) and (4.4) in eqns. (3.7) and (3.8) we have

$$\xi = \frac{1}{M_1},\ j = \frac{kM_3}{M_1}. \qquad \ldots(4.5)$$

And therefore, the magnetic field by using (4.3) and (4.4) in (4.1) we have

$$H_1 = -k\left[\frac{2M_1(x + M_4 y)}{(x^2 + y^2)} + \frac{M_3}{2M_1} y\right]$$

$$H_2 = k\left[\frac{2M_1(M_4 x - y)}{(x^2 + y^2)} + \frac{M_3 x}{2M_1}\right]. \qquad \ldots(4.6)$$

Thus the magnetic field **H** is determined, apart from an arbitrary constant factor and it is to be noted that H becomes infinite at the stagnation point of the flow[4]. The magnetic field strength decays continuously as the radial distance from the central axis varies. Using (4.5), (2.9), (2.10), and integrability condition for h, we have,

$$x\frac{\partial\omega}{\partial y} - y\frac{\partial\omega}{\partial x} = 0. \qquad \ldots(4.7)$$

Equation (4.7) is the differential equation of circle whoes radius is always less than unity. The most general solution of (4.7) is,

$$\omega = y^2\,{}_1F_0\left[-1; -; -\frac{x^2}{y^2}\right] = \sum_{n=0}^{\infty}\frac{(-1)_n(-1)^n x^{2n} y^{2-2n}}{n!} \qquad \ldots(4.8)$$

where ${}_1F_0$ is Gauss' Hypergeometric function. Taking the most particular solution when $n = 1$ we have

$$\omega = c(x^2 + y^2) = cr^2\ c \neq 0. \qquad \ldots(4.9)$$

Finally employing (4.3), (4.4), (4.5), (4.6) and (4.9) in eqns. (3.13) and (3.14) gives

$$p = h - \tfrac{1}{2}\rho q^2$$

$$p(x, y) = \frac{\rho}{8M_1^2} r^2 + 2M_3\mu k^2 \tan^{-1}\left(\frac{x}{y}\right) - \frac{M_3^2\mu k^2}{4M_1^2} r^2$$

$$- M_3 M_4 \mu k^2 n r^2 - \frac{cr^4}{2M_1} + D \qquad \ldots(4.10)$$

where D is an arbitrary constant.

The equation represents that the pressure at any point varies as the radial distance from the central axis.

5. Radial Flow

Another simple solution of eqns. (3.14).

$$\left.\begin{aligned} L(u, v) &= N_1 \tan^{-1}\frac{v}{u} + N_2 \\ \alpha(u, v) &= \cot^{-1} N_3 q^2. \end{aligned}\right\} \qquad \ldots(5.1)$$

From eqn. (3.1)

$$x = \frac{\partial L}{\partial v} y = - \frac{\partial L}{\partial u}.$$

This equation express again as more conveniently.

$$u(x, y) = N_1 x/r^2, \; v(x, y) = N_1 \, y/r^2. \qquad \ldots(5.2)$$

This represents purely radial flow, and velocity profile is thus the arc of a rectangular hyperbola. From eqns. (2.7), H is given by,

$$\left.\begin{aligned} H_1(x, y) &= \frac{k}{N_1}\left[\frac{N_1^2 N_3 x - y r^2}{r^2}\right], \\ H_2(x, y) &= \frac{k}{N_1}\left[\frac{N_1^2 N_3 y + x r^2}{r^2}\right] \end{aligned}\right\}. \qquad \ldots(5.3)$$

Now integrability condition for h yields

$$y \frac{\partial \omega}{\partial y} + x \frac{\partial \omega}{\partial x} = 0. \qquad \ldots(5.4)$$

The most general solution of (5.4) is

$$\omega = {}_2F_1\left[a, b, c; \frac{y}{x}\right] = \sum_{n-0}^{\infty} \frac{(a)_n (n)_n Y^n x^{-n}}{(c)_n \; n!}, \; x \neq 0 \qquad \ldots(5.5)$$

where

$${}_2F_1[a, b, c; Z] = 1 + \frac{ab}{c} \frac{Z}{1} + \frac{a(a-1)b(b+1)}{c(c+1)} \frac{Z^2}{2}$$

$$+ - + - + -$$

is Gauss hypergeometric function, a, b, c are const. ($c \neq 0$). Taking most particular solution for $n = 1$

$$\omega = D_1 \, y/x, \; x \neq 0 \qquad \ldots(5.6)$$

D_1 is an arbitrary non-zero constant.

Then

$$p(x, y) = \frac{\mu k^2}{N_1^2}\left[2N_1^2 \; N_3 \tan^{-1} \frac{y}{x} - r^2\right] - \frac{N_1^2}{2r^2} - D_1 k \ln \frac{x^2}{r^2} + D_3 \qquad \ldots(5.7)$$

D_2 is an arbitrary const.

Summing up: A variably inclined plane rotating MHD flow problem with the family of stream lines and magnetic lines given by $y/x = \text{const.}$

$(x^2 + y^2) - 2N_1^2 \; N_3 \tan^{-1}(y/x) = \text{const.}$ having variable angle $\alpha\,(x,y) = \cot^{-1}$ $(N_3 \, N_1^2 \, / r^2)$ between them.

6. Hyperoblic Flow

In this case we take

$$L\,(u, v) = Au^2 + Bv^2 \qquad \ldots(6.1)$$

where A and B are two non zero unequal real number using (6.1) in (3.6) the partial differential eqn. in (u, v) is given by

$$Bv\,(u^2 + v^2) \frac{\partial}{\partial u} \cot\alpha - Au\,(u^2 + v^2) \frac{\partial}{\partial v} \cot\alpha + (2Auv - 2Buv) + (B - A)(v^2 - u^2) = 0. \qquad \ldots(6.2)$$

The general solution of (6.2) is

$$\cot\alpha = \frac{(A - B)\,uv}{Au^2 + Bv^2} + B\,(u^2 + v^2)\,F\,(Au^2 + Bv^2) \qquad \ldots(6.3)$$

where $F\,(Au^2, Bv^2)$ is an arbitrary function of its argument. Using expression $L\,(u, v)$, $\alpha\,(u, v)$ from (6.1)-(6.3) in eqns. (3.15)-(3.19) we have,

$$J\,(u, v) = \frac{1}{4AB}, \; \xi = \frac{A + B}{2AB}$$

$$j\,(u, v) = k\left(\frac{A + B}{2A}\right) F\,(Au^2 + bv^2) + kB\,(u^2 + v^2)\,F^1\,(Au^2 + Av^2) + \frac{k\,(B - A)\,uv}{(Au^2 + Bv^2)} \qquad \ldots(6.4)$$

$$Q_1\,(u, v) = Q_2\,(u, v) = 0.$$

We now use (6.1), (6.3) and (6.4) in (3.14) and obtain the equation that A, B and F $(Au^2 + Bv^2)$ must satisfy so that assumed $L\,(u, v)$ is the Legendre transform and derived $\alpha\,(u, v)$ is the transformed variable angle. This equation is

$$2AB\,(u^2 + v^2)\;F'' + 2B\,(B - A)\,uvF\,F' + \left[\frac{(A + B)\,(Au^2 + Bv^2)}{} + \frac{2AB\,(u^2 - v^2)}{Au^2 + Bv^2}\right] F' + (A - B)\left[\frac{Au^2 - Bv^2}{(Au^2 + Bv^2)^2}\right] F + \frac{2A\,(A - B)\,uv}{(Au^2 + Bv^2)^3} = 0 \qquad \ldots(6.5)$$

where F', F'' are the first and second derivatives of F with respect to its argument, one of the solution of (6.5) is

$$F(Au^2 + Bv^2) = E(Au^2 + Bv^2)^m \qquad \text{...(6.6)}$$

where E, m are two arbitrary real numbers. Substituting in (6.5) we get

$$2E^2 MB(B - A)uv(Au^2 + Bv^{22})^{2m-1} + [2EM^2 AB(u^2 + v^2) + 2EB(B - A)v^2](Au^2 + Bv^2)^{m-2} + E(AM + BM + A - B)(Au^2 + Bv^2)^{m-1} + 2A(A - B)uv(Au^2 + Bv^2)^{-3} = 0. \qquad \text{...(6.7)}$$

Since this equation is identically satisfied if and only if $m = -1$ and $E^2 = -A/B$ if follows that

(i) one of the two unequal numbers A, B is to be positive and other is negative

(ii) $F(Au^2 + Bv^2) = \pm E^2(Au^2 + Bv^2)^{-1}$ are only solution of (6.5), taking $E > 0$ and

$$A = a^2 > 0; \; B = -b^2 > 0; \; a, b \in R$$

we have

$$L(u, v) = a^2 u^2 - b^2 v^2, \; (u,v) = \cot^{-1}\left[\frac{av - bu}{au + bv}\right] \qquad \text{...(6.8)}$$

following the previous flow, we find that

$$u(x, y) = -y/2a^2, \; v(x, y) = -x/2b^2 \qquad \text{...(6.9)}$$

$$H_1(x, y) = \frac{2k\,ab^2}{by + ax}, H_2^2(x, y) = -\frac{2ka^2b}{by + ax}. \qquad \text{...(6.10)}$$

Another rotating variably inclined flow problem when c 0 and

$$A = a^2; \; B = -b^2 < 0; \; a, b \in R \qquad \text{...(6.11)}$$

corresponds to the solution set

$$L(u, v) = a^2 u^2 - b^2 v^2 \qquad \text{...(6.12)}$$

$$\alpha(u, v) = \cot^{-1}\frac{av + bu}{au - bv}$$

of eqns. (3.13)–(3.15) using the solution set we find the stream lines $a^2 x^2 - b^2 y^2 =$ const. and magnetic lines $ax - by =$ const. from rotating variably inclined flow problem with

$$\alpha(x, y) = \cot^{-1}\frac{a^3 x + b^3 y}{ab(by - ax)} \qquad \text{...(6.13)}$$

solution of this problem is

$$u(x, y) = \frac{y}{2a^2}, \; v(x, y) = \frac{x}{2b^2} \qquad ...(6.14)$$

$$H_1(x, y) = \frac{2kab^2}{ax - by}, \; H_2(x, y) = \frac{2ka^2 b}{ax - by}$$

and integrability condition for h yields

$$\omega = D_3\, y/x, \; D_3 \neq 0. \qquad ...(6.15)$$

7. Applications

In general, exact solutions in magnetohydrodynamics are rare. In this paper Legendre approach is employed to obtain exact solutions for steady, variably inclined plane rotating MHD flow problems. Equations (2.9) and (2.10) are in countered in meteorological problems in which ωu, and ωv represent the components of acceleration produced by Coriolis force owing the rotation of the earth. First time we obtain the general value of Coriolis parameter for radial, vortex and hyperbolic flows. Bloomer *et al.*[5] have studied the problems of designing a channel in which there in uniform flow at the mouth, and such that pressure gradient falls continuously. If in practice we, construct the boundaries from suitable conducting material and provided that the fluid itself is suitable chosen then the MHD channel flows produced will indeed be such that the velocity and magnetic field are orthogonal everywhere at least to a fair degree of approximation. Power and Walker[4] have shown that invariably inclined MHD flows with orthogonal velocity and magnetic fields, magnetic intensity can be determined for any small value of k^2. The length of the channel in which the flow exists depends on the magnitude of k^2, and for $k^2 = 0$ we find the channel flow of a gas with zero maganetic field.

The effect of rotation are to produced a cross channel drift of fluid which causes the flow in the test section to be inclined to the channel exis by a small angle[6,8], where $\tan \beta = (2L)^{-1} (\nu/\xi)^{1/2}$ where L is the length of channel.

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HALL EFFECTS ON UNSTEADY MHD FREE AND FORCED CONVECTION FLOW IN A POROUS ROTATING CHANNEL

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We make an initial value investigation of an unsteady combined free and forced convection flows in porous rotating channel. Both the porous walls execute non-torsional oscillations in their own plane and in addition they are maintained at a constant temperature gradient. The solutions for the velocity, temperature ditributions and the shear stresses are obtained in closed form by using Laplace transform technique. The structure of the associated boundary layers are discussed for small ($\omega \ll 1$), intermediate ($\omega = O(1)$) and high frequency ($\omega \gg 1$) of oscillations.

1. INTRODUCTION

The study of hydromagnetic viscous rotating conducting flows has drawn the attention of many researchers due to its applications in cosmical and geophysical fluid dynamics. In investigating the interplay of the Coriolis force, magnetic force and viscous forces, Nanda and Mohanty[5] have considered the hydromagnetic flow in a rotating channel formed by two infinite horizontal plates under the action of a constant pressure gradient. Assuming the flow to be steady, fully developed they have shown that when the applied magnetic field and rotation are weak, the effect of the magnetic field on the flow in the direction normal to the pressure gradient is unaffected by rotation. Also for a strong magnetic field they observed modified Ekman layers near the boundaries whose thickness is inversely proportional to the magnetic field and the shear stresses at the plates always decrease with increase in the magnetic parameter. Mohan[3] has studied the free convection effects for a similar configuration. It is found that when the Grashoff number G is large, the fluid in the vicinity of the two plates move in the opposite directions and the flow separation takes place only at the lower plate.

In the above mentioned problems the forced flow in the rotating channel is generated due to a constant pressure gradient and the interaction of the magnetic field with or without the effects of free convection current on this forced flow, is investigated. The channel flow problems where in the forced flow is generated by the torsional or non-torsional oscillations of one or both boundaries is of some interest in recent times since these problems enable us to find the growth and development of boundary layers

associated with the hydromagnetic flows occurring in Geophysical phenomena. Claire Jacobs[1] has studied the transient effects in a viscous rotating fluid by considering the small amplitude torsional oscillations of disks. This problem has been extended to hydromagnetics by Murthy[4]. Debnath[2] has discussed an unsteady hydrodynamic and hydromagnetic boundary layer flow in rotating viscous fluid due to oscillations of the boundary including the effects of uniform pressure gradient and uniform suction. Taking Hall current effects into consideration Rao *et al.*[6] have studied the free convection in rotating channel.

In this paper we make an initial value investigation of an unsteady combined free and forced convection flow in a rotating channel bounded by two porous, insulated walls. Both the porous walls execute non-torsional oscillations in their own plane and in addition they are maintained at a constant temperature gradient in some arbitrary direction parallel to the plane of the walls. The solutions for the velocity, temperature distributions and the shear stresses are obtained in closed form by using Laplace transform technique. The structure of the associated boundary layers are discussed for small ($\omega \ll 1$), intermediate ($\omega = O(1)$) and for high frequency ($\omega \gg 1$) of oscillations.

2. Formulation and Solution

We consider the unsteady flow of an incompressible viscous, electrically conducting fluid bounded by parallel, non-conducting porous walls under a uniform transverse magnetic field H_0. In the equilibrium state both the walls and the fluid rotate with the same angular velocity and are maintained at constant temperature. At $t > 0$, the fluid is driven by a constant pressure gradient in an arbitrary direction parallel to the walls and in addition the disks are executing harmonic oscillations with different frequencies. The walls are cooled or heated by a constant temperature gradient in a direction parallel to the plane of the walls. We choose a cartesian coordinate system $O(x', y', z')$ such that the disks are at $z' = -L$ and $z' = L$ and the z'-axis considering with the axis of rotation of the walls. Taking the Hall currents into account the unsteady hydromagnetic boundary layer equations of motion with respect to a rotating frame moving with angular velocity Ω, in the absence of any input electric field, in the component form are

$$u'_t + W_0 u'_z - 2\Omega v' = -\frac{1}{\rho} p'_x + \nu u'_{zz} + \frac{\sigma \mu_e H_0}{\rho} J_y \qquad \text{...(2.1)}$$

$$v'_{t'} + W_0 v'_z + 2\Omega u' = -\frac{1}{\rho} p'_y + \nu v'_{zz} - \frac{\sigma \mu_e H_0}{\rho} J_x \qquad \text{...(2.2)}$$

$$0 = -\frac{1}{\rho} p_z - g(1 - \beta(T' - T_0)). \qquad \text{...(2.3)}$$

Magnetic Induction equations are

$$J_x + m J_y = (\sigma \mu_e H_0) v' \qquad \text{...(2.4)}$$

$$J_y - m J_x = -(\sigma \mu_e H_0) u' \qquad \text{...(2.5)}$$

and the energy equation is

$$\left(\frac{\partial}{\partial t} + u' \frac{\partial}{\partial x'} + W_0 \frac{\partial}{\partial z'} + v' \frac{\partial}{\partial y'}\right) (T' - T_0) = \alpha \frac{\partial^2}{\partial z'^2} (T' - T_0) \quad ...(2.6)$$

where u', v' are the velocity components along x' and y' directions respectvely, p' the pressure including the centrifugal force, ρ the density, μ_e the magnetic permeability, σ the electrical conductivity of the fluid, α the thermal diffusivity, J_x and J_y are the components of the current density and m the Hall parameter. In writing equations (2.4) and (2.5) the ion slip, the electron pressure gradient and the thermo-electric effects are neglected

Combining eqns. (2.1)—(2.5) we obtain

$$F'_{t'} + W_0 \; F'_{z'} + \; 2i\,\Omega\, F' = - \; \frac{1}{\rho} \, (p'_{x'} + i \; p'_{y'}) + \nu \; F'_{z'z'}$$

$$- \; \frac{\sigma\mu_e^2 \; H_0^2}{\rho\,(1 - im)} \, F' \qquad ...(2.7)$$

where

$$F' = (u' + i\, v').$$

Integrating (2.3) we get

$$p'/\rho = - \; gz' + \beta G \int (T' - T_0)\, dz' + \phi\, (\xi', \bar{\xi}')\, H\,(t') \qquad ...(2.8)$$

where

$$\xi' = x' - iy', \; \bar{\xi}' = x' + iy' \; H\,(t')$$

is the Heaviside function.

Using (2.3), eqn. (2.7) can be written as

$$\frac{\partial}{\partial z'} \, (F'_{t'} + 2i\Omega\, F' + W_0 \; F'_{z'z}) = \nu \; F'_{z'z'} - \; 2\,\beta g\,(T' - T_0)\frac{\partial}{\partial \xi}\,(T' - T_0)$$

$$- \; \frac{\sigma\,\mu_e^2 \; H_0^2}{\rho\,(1 - im)} \, F'. \qquad ...(2.9)$$

Since $F' = F\;(z', t')$, equation (2.9) is valid if the temperature distribution is of the form

$$(T' - T_0) = (\alpha_1\, x' + \beta_1 y')\, H\,(t') + \theta'_1\,(z', t') \qquad ...(2.10)$$

where α_1 and β_1 are the gradients of the temerature along $O\;(x', y')$ directions respectively and $\theta'_1\;(z', t')$ is an arbitrary function of z' and t'. We take

$$T_0 + \alpha_1\, x' + \beta_1\, y' + \theta'_{1w_1} \text{ and } T_0 + \alpha_1\, x' + \beta_1\, y' + \theta'_{1w_2}$$

to be the temperatures of the lower and upper disks respectively for $t > 0$.

From (2.7), (2.8) and (2.10) we obtain

$$F'_{t'} + W_0 F'_{z'} + 2i\Omega F' - \nu F'_{z'z'} + \frac{\sigma \mu_e^2 H_0^2}{\rho (1 - im)} F' = \beta g \, Az' \, H(t') - D \qquad \text{...(2.11)}$$

where

$$D = \frac{\partial}{\partial \xi} \phi (\xi', \bar{\xi}), \; A = \alpha_1 + i\beta_1.$$

Introducing the dimensionless variables $(z, F, t, \omega, \theta)$ as

$$z = z'/L, \; F = F' \, L/\nu, \; t = t' \, (\nu/L^2), \; \omega = \omega' \, (L^2/\nu),$$

$$\theta = \beta g \, L^3 \, (\theta'_1 - \theta'_{1w_1})/\nu^2$$

the governing equations (2.6) and (2.11) reduce to

$$F_t + R_0 F_z - F_{zz} + [M^2/(1 - im) + 2i \, E^{-1}] \, F + GzH(t) = R \qquad \text{...(2.12)}$$

$$P(\theta_t + R_0 \, \theta_z) = \theta_{zz} - 2P \, (G_r u + G_l \, v) \, H(t) \qquad \text{...(2.13)}$$

where

$R_0 = w_0 L/\nu$ (Cross flow Reynolds number)

$M^2 = \dfrac{\sigma \mu_e^2 H_0^2 L^2}{\rho\nu}$ (Hartmann number)

$m = \omega_e \tau_e$ (Hall parameter)

$P = \alpha/\nu$ (Prandtl number)

$\left.\begin{aligned} G_r &= \beta g \, \alpha_1 \, L^4/\nu^2 \\ G_l &= \beta g \, \beta_1 \, L^4/\nu^2 \end{aligned}\right\}$ (Grashoff numbers)

$R = (-L^3/\nu^2) \, D$

$G = G_r + i \, G_l.$

The boundary conditions in the non-dimensional form are

$$F(z, t) = \begin{cases} a \exp(i\omega_1 t) & \text{on } z = +1 \\ b \exp(i\omega_2 t) & \text{on } z = -1 \end{cases} \qquad \text{...(2.14)}$$

$$\theta(z, t) = \begin{cases} 0 & \text{on } z = +1 \\ \beta g \, L^3 \, (\theta'_{1w_2} - \theta'_{1w_1})/\nu^2 = \theta_0 \text{ (say) on } z = -1 \end{cases}$$

and the initial conditions are

$$
\left.\begin{aligned}
F(z, t) &= 0 \\
\theta(z, t) &= 0.
\end{aligned}\right\} \quad \text{for } t \leqslant 0.
$$

By taking Laplace transforms, the governing equations (2.12) and (2.13) reduce to

$$F_{zz} - R_0 F_z - (\lambda^2/E) F = \frac{(Gz - R) E}{s} \qquad \text{...(2.16)}$$

$$\theta_{zz} - P(R_0 \theta_z + s\theta) = 2P(G_r u + G_l v) \qquad \text{...(2.17)}$$

where

$$\lambda^2 = s E + 2i + (M^2 E/(1 - im)$$

and the boundary conditions are

$$F(z, s) = \begin{cases} a/(s - i\omega_1) & \text{on } z = +1 \\ b/(s - i\omega_2) & \text{on } z = -1 \end{cases} \qquad \text{...(2.18)}$$

$$\theta(z, s) = \begin{cases} 0 & \text{on } z = +1 \\ \theta_0/s & \text{on } z = -1. \end{cases} \qquad \text{...(2.19)}$$

The solution of (2.16) subject to (2.18) is

$$F(z, s) = \exp(R_0 z/2)[A \cosh(pz) + B \sinh(pz)] + \frac{(R - Gz) E}{s\lambda^2} + \frac{G R_0 E^2}{s\lambda^4} \qquad \text{...(2.20)}$$

where

$$p = \sqrt{(R_0/2)^2 + \lambda^2/E}$$

$$2A \cosh(p) = (a/(s - i\omega_1)) \exp - (R_0/2 + (b/(s - i\omega_2)) \exp(R_0/2)$$
$$- \frac{2GE}{s\lambda^2} \sinh - (R_0/2) - \cosh(R_0/2) \frac{2E(R_0 EG + R\lambda^2)}{s\lambda^4}$$

$$2B \sinh(p) = (a/(s - i\omega_2)) \exp - (R_0/2) - (b/(s - i\omega_2)) \exp(R_0/2)$$
$$+ \frac{2GE}{s\lambda^2} \cosh(R_0/2) + \frac{2E(R_0 EG + R\lambda^2)}{s\lambda^4}. \qquad \text{...(2.21)}$$

The function $\bar{F}$ has simple poles at $s = 0, i\omega_1, i\omega_2, -s_{n1}, -s_{n2}$ and a double poles at $s = -\lambda_1$,

where

$$s_{n_1} = 0.25\,[4\,M^2_{j(1-im)} + R_0^2 + 4\,n^2\pi + 8\,iE^{-1}]$$

$$s_{n2} = 0.25\,[4M^2_{j(1-im)} + R_0^2 + (2n+1)^2 + 8\,iE^{-1}].$$

Using the caculus of residues, inversion of $F(z, s)$ is given by

$$\begin{aligned} F(z,t) &= 0.5\,a \exp(i\omega_1 t + R(z-1)/2)\left[\frac{\cosh(\lambda_2 z)}{\cosh \lambda_2} + \frac{\sinh(\lambda_2 z)}{\sinh \lambda_2}\right] \\ &\quad + 0.5\,b \exp(i\omega_2 t + R_0(z+1)/2) \\ &\quad \times \left[\frac{\cosh(\lambda_3 z)}{\cosh \lambda_3} - \frac{\sinh(\lambda_3 z)}{\sinh \lambda_3}\right] - \exp(R_0 z/2) \\ &\quad \times \left[a_3 \frac{\cosh(\lambda_4 z)}{\cosh \lambda_4} - a_4 \frac{\sinh(\lambda_4 z)}{\sinh \lambda_4}\right] + \frac{(R - Gz)E}{\lambda_1} \\ &\quad + \frac{G R_0 E^2}{\lambda_1^2} + \theta(\exp(-s_{n_1} t)) + \theta(\exp(-\lambda_1 t)) \end{aligned} \qquad \text{...(2.22)}$$

where

$$\lambda_1^2 = 2i + M^2E/(1 - im)$$

$$\lambda_2^2 = \left(\frac{R_0^2}{4} + \frac{M^2}{1+m^2}\right) + i\left(\omega_2 + 2E^{-1} + \frac{mM^2}{1+m^2}\right)$$

$$\lambda_3^2 = \left(\frac{R_0^2}{4} + \frac{M^2}{1+m^2}\right) + i\left(\omega_2 + 2E^{-1} + \frac{mM^2}{1+m^2}\right)$$

$$\lambda_4^2 = \left(\frac{R_0^2}{4} + \frac{M^2}{1+m^2}\right) + i\left(2E^{-1} + \frac{mM^2}{1+m^2}\right)$$

$$a_3 = [G\,R_0\,E^2 \cosh(R_{0/2}) + \lambda_1 E(R\cosh(R_{0/2}) + G\sinh(R_{0/2})]\lambda_1^2$$

$$a_4 = [G_0\,R_0\,E^2 \sinh(R_{0/2}) + \lambda_1 E(G\cosh(R_{0/2}) + \sinh(R_{0/2})]\,\lambda_1^2.$$

The non-dimensional shear stress τ_x and τ_y at the wall are given by

$$\begin{aligned} (\tau_x + i\tau_y)_{z=-1} &= 0.5\,a\,\lambda_2 \exp(i\omega_1 t - R_0)(1/a_5 - a_5) \\ &\quad + 0.5\,b \exp(i\omega_2 t)(R_0 - \lambda_3(1/a_6 + a_6)) \\ &\quad - a_3 \exp(-R_{0/2})(R_{0/2} - \lambda_4 a_7) \\ &\quad + a_4 \exp(-R_{0/2})(R_{0/2} + \lambda_{4/a_7}) - GE/\lambda_1 \end{aligned}$$

and

$$(\tau_x + i\tau_y)_{z=1} = 0.5\,a \exp(i\omega_1 t)(R_0 + \lambda_2(a_5 + 1/a_5))$$

$$+\ 0.5\, b\, \lambda_3 \exp(i\omega_2 t + R_0)(a_6 - 1/a_6)$$
$$-\ a_3 \exp(R_{0/2}(R_{0/2}) + \lambda_4 a_7)$$
$$+\ a_4 \exp(R_{0/2}(R_{0/2}) + \lambda_4/a_7) - GE/\lambda_1$$

where

$$a_5 = \tanh(\lambda_2),\ a_6 = \tanh(\lambda_3),\ a_7 = \tanh(\lambda_4)$$

substituting $\bar{F}$ in transformed energy equation (2.17) and using invese Laplace transform, we obtain the Temperature distribution θ.

3. Discusion of the Associated Boundary Layers

We shall discuss the interplay between the Hall parameter m, the Hartmann number M and the Ekman number E in determining the time required for the decay of the transient terms in the solution.

From (2.22), it follows that the transient velocity decays in dimensional time of order $1/(\pi + R_0^2/4 + M^2/(1 + m^2))$ which implies that the decay time of the transient velocity is less than the decay time in the absence of any magnetic field. Also, the decay time increases with increasing m and decreases with R_0

After the decay of transient terms in (2.22) the steady oscillatory distribution is given by

$$u + iv = a/2 \exp(i\omega_1 t + R_0(z - 1)/2)\left[\frac{\cosh(\lambda_2 z)}{\cosh \lambda_2} + \frac{\sinh(\lambda_2 z)}{\sinh \lambda_2}\right]$$
$$+ b/2 \exp(i\omega_2 t + R_0(z + 1)/2)\left[\frac{\cosh(\lambda_3 z)}{\cosh \lambda_3} + \frac{\sinh(\lambda_3 z)}{\sinh \lambda_3}\right]$$
$$+ \exp(R_0 z/2\, [\lambda_1 GE + GR_0E^2)\cosh(R_0/2) - R\lambda_1 E \sinh(R_0/2)]$$
$$\times\ \frac{\cosh(\lambda_4 z)}{\cosh \lambda_4} + [(GR_0E^2 + \lambda_1 GE)\sinh(R_0/2)$$
$$+ (\lambda_1 RE)\cosh(R_0/2)] \times \frac{\sinh(\lambda_4 z)}{\sinh \lambda_4} + \frac{(R - Gz)E}{\lambda_1} + \frac{GR_0E^2}{\lambda_1^2}. \quad \ldots(3.1)$$

This consists of two parts : (i) The hyperbolic terms in z which gives rise to layers of thickness of order

$$[(a_1^2 + a_2^2)^{1/2} + a_1]^{-1/2} \text{ if } \omega_1 < \omega_2$$

or

$$[(a_1^2 + b_1^2)^{1/2} + a_1]^{-1/2} \text{ if } \omega_1 > \omega_2$$

or

$$[(a_1^2 + b_2^2)^{1/2} + a_2]^{-1/2} \text{ if } \omega_1 = \omega_2$$

where

$$a_1 = R_0^2/4 + M^2/(1 + m^2)\ ;\ b_1 = 2E^{-1} + m\,M^2/(1 + m^2)$$

$$a_2 = \omega_1 + b_1 \qquad ;\ b_2 = \omega_2 + b_1$$

near the disks through which the motion of the disks is communicated to fluid. Thus the Lorentz-Coriolis force balance gives rise to two types of layers described viz. Stokes-Ekman-Hartmann layer and an Ekmann layer respectively.

(ii) The remaining terms represent an interior flow. We shall consider the following cases of the frequency of oscillations.

(i) *Low Frequency* ($\omega \ll 1$)

When $m \gtrsim 1$ we get a boundary layer of thickness of order

$$\left(\frac{R_0^2}{4} + M^2\right)^{-1/2} \text{ if} \left(\frac{R_0^2}{4} + M^2\right) \omega \gg E^{-1}$$

and

$$O\,(E^{-1/4}) \text{ if } \left(\frac{R_0^2}{4} + M^2\right) \omega \ll 2\,E^{-1}.$$

In this case the thickness of the boundary layer does not mainly depend on m.

(ii) *Intermediate Frequency* ($\omega \sim O\,(1)$)

Since $\omega \sim O\,(1)$, $|\ \omega + 2E^{-1} + mM^2/(1 + m^2)\ | \sim O\,(1)$ then the thickness of boudary layer is the order

$$\left(\frac{R_0^2}{4} + \frac{M^2}{1 + m^2}\right)^{-1/2} \text{ if} \left(\frac{R_0^2}{4} + \frac{M^2}{1 + m^2}\right) \gg \left|\ \omega + 2E^{-1} + \frac{mM^2}{1 + m^2}\ \right|$$

and

$$0\left(\ \left|\ \omega + 2E^{-1} + \frac{m\,M^2}{1 + m^2}\ \right|^{-1/2}\ \right) \text{ if} \left(\frac{R_0^2}{4} + \frac{M^2}{1 + m^2}\right) << \left|\ \omega + 2E^{-1} + \frac{mM^2}{1 + m^2}\ \right|$$

Further, the thickness reduces to

$$|\ \omega + 2E^{-1}\ |^{1/2} \text{ if } \omega \gg \frac{m\,M^2}{1 + m^2}\cdot$$

and to

$$\left|\ 2E^{-1} + \frac{mM^2}{1 + m^2}\ \right|^{-1/2} \text{ if } \omega \gg \frac{mM^2}{1 + m^2}$$

(iii) *High Frequency* ($\omega \gg 1$)

When $M \sim 0(1)$, $m < 1$ and for all E the thikness of the boundary layer is

$$\left[\frac{R_0^2}{4} + M^2 + \left(\left(\frac{R_0^2}{4} + M^2\right)^2 + (\omega + 2E^{-1} + M^2)^2\right)^{1/2}\right]^{-1/2}$$

In particular, if $\omega \gg M$, E^{-1} and R_0 the thickness is of order $(\omega^{-1/2})$. However, if $\omega \gg (2E^{-1} + M^2)$ the thickness is of order

$$\omega^{-1/2}\left(\frac{R_0^2/4 + M^2}{2\omega}\right).$$

It is to be noted that in any case for large ω the thickness of the boundary layer increases with an increase in Hall parameter m for a fixed M while it decreases with an increase in M for a fixed m.

4. Discussion of the Velocity Field and Stress

Rewritting the steady oscillatory velocity distribution (3.1) in exponential form and keeping in view the behaviour of each of the terms in it we find that an increase in Hall parameter m through values lower than 2, increases the magnitude of velocity components u and v for fixed values of G, M, ω and R_0 whereas for $m > 2$ their magnitudes decrease with an increase in m. Also their magnitudes increase with an $|G|$ or R_0 keeping the other parameters fixed. For $G > 0$, u may be found to attain its maximum near the lower wall while it attains its maximum near the upper wall for $G < 0$. The magnitudes of u and v for $\omega_1 > \omega_2$ are in general found to be larger than their corresponding magnitudes for $\omega_1 < \omega_2$.

The stress τ_x increases (decreases) at the upper (lower) wall the stress τ_y decreases at both the walls for an increase in m or R_0. Also the values of τ_x at the upper wall for $\omega_1 > \omega_2$ is greater than its corresponding values for $\omega_1 < \omega_2$. At the lower wall a reversed effect is observed.

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EIGENFUNCTION EXPANSION METHOD TO THERMOELASTIC AND MAGNETO-THERMOELASTIC PROBLEMS

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The fundamental equations of the general one dimensional problems of thermoelasticity and magneto-thermoelasticity have been written in the form of an inhomogeneous vector-matrix differential equation and solved in the Laplace transform domain by the eigen-function expansion method. It has been pointed out that a broad class of problems in mechanics can be solved by this approach. Applications to problems pertaining to an infinite medium due to instantaneous heat sources are presented and compared with existing literature. Finally, the solution for the space time domain has been made by numerical methods and the results are analysed.

Nomenclature

The nomenclature is as follows :

$\mathbf{H}$ = magnetic field

$\mathbf{B}$ = magnetic induction vector

$\mathbf{j}$ = current density vector

$\mathbf{E}$ = electric field vector

$\mathbf{u}$ = mechanical displacement vector

$\mathbf{H}$ = $\mathbf{H}_0 + \mathbf{h}$, $\mathbf{H}_0 = [H_1, H_2, H_3]$, $\mathbf{h} = [h_x, h_y, h_z]$

$\mathbf{H}_0$ = primary magnetic field

$\mathbf{h}$ = perturbation of $\mathbf{H}$

τ_{ij} = mechanical stress tensor

e_{ij} = strain tensor

T = differential temperature distribution

μ_e = magnetic permeability

ν_H = $\dfrac{1}{\eta\mu e}$, magnetic viscosity

η = electrical conductivity

c = velocity of light in vacuum

λ, μ = Lame elastic constants

ρ = density

α = coefficient of linear thermal expansion

T_0 = reference temperature of the body when unstressed

C_v = specific heat at constant volume

k_1 = $\dfrac{k}{\rho C_v}$, thermal diffusivity

k = thermal conductivity

Q = heat source

F = body force component

$\delta(t)$ = Dirac delta function

$$c_1^2 = \frac{\lambda + 2\mu}{\rho}, \quad b_1 = \frac{\beta}{\rho C_v}, \quad \beta = (3\lambda + 2)\alpha$$

$$d_1 = \frac{k_1}{T_0 \ \rho C_v c_1^2}, \quad a_1 = \frac{\beta T_0}{\lambda + 2\mu}$$

$$b = \frac{H_3^2}{\lambda + 2\mu}, \quad d = \frac{4\pi k_1}{c_1^2 \ \nu H}$$

$$\epsilon = \frac{a_1 \beta}{\rho C_v}.$$

Introduction

In recent papers Bahar and Hetnarski[1-3], have extended the state space approach developed hy Bahar in boundary value problems[4] and heat conduction[5] to problems of coupled thermoelasticity. Das *et al.*[6] applied the state space approach to magneto-thermoelasticity as a further extension of Bahar and Hetnarski[1]. Das *et al.*[7-10], also applied eigenvalue approach to the problems of thermoelasticity and magnetothermo-elasticity. But in all these papers the authors have neglected the body forces in the equations of motion, the heat sources in the equation of heat conduction etc. As a result the vector-matrix differential equation involved becomes homogeneous. Recently Das *et al.*[11] have solved the problems of thermoelasticity and magneto-thermoelasticity by taking into account of the body forces and heat sources, and writing the equation in

an inhomogeneous vector-matrix differential equation. The solution has been made by state space approach.

In this paper we have attempted to solve the problem[11] by the eigenfunction expansion method. Theory for the solution of an inhomogenous vector-matrix differential equation has been presented and for its applications we have chosen two particular problems of elasticity coupled with (i) a thermal field and (ii) a thermomagnetic field.

Theory

Consider the vector-matrix differential equation

$$\frac{d\mathbf{v}}{dx} = \mathbf{A}\ \mathbf{v} + \mathbf{f}\ (x) \qquad \ldots(1)$$

with the condition

$$\mathbf{v}\ (x_0) = \mathbf{c} \qquad \ldots(2)$$

where $\mathbf{A}$ is an $n \times n$ constant real matrix, $\mathbf{c}$ is a given constant real n-vector and $\mathbf{f}$ is a real n-vector function.

Let

$$\mathbf{v} = \mathbf{X}\ e^{\lambda x} \qquad \ldots(3a)$$

be a solution of the homogeneous equation

$$\frac{d\mathbf{v}}{dx} = \mathbf{A}\ \mathbf{v} \qquad \ldots(3b)$$

where λ is a scalar and $\mathbf{X}$ is an n-vector independent of x. Substituting (3a) in (3b) we get

$$(\mathbf{A}\ \mathbf{X} - \lambda\mathbf{X})\ e^{\lambda x} = 0 \Rightarrow \mathbf{A}\ \mathbf{X} - \lambda\mathbf{X} = 0 \Rightarrow \mathbf{A}\mathbf{X} = \lambda\mathbf{X}.$$

This may be interpreted that λ is an eigenvalue of the matrix $\mathbf{A}$ and $\mathbf{X}$ the corresponding right eigenvector.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$, be n distinct eigenvalues of the matrix $\mathbf{A}$ and $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be the corresponding right eigenvectors of the matrix $\mathbf{A}$. Then the vectors $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are linearly independent and so they form a basis of the space Γ^n, where Γ denotes the field of complex numbers. We can find scalars $b_1, b_2, \ldots, b_n$ such that

$$\mathbf{c} = b_1\ \mathbf{X}_1 + b_2\ \mathbf{X}_2 + \ldots + b_n\ \mathbf{X}_n.$$

Choose

$$c_i = b_i\ e^{-\lambda_i x_0} \quad (i = 1, 2, \ldots, n).$$

Let

$$\mathbf{u}\ (x) = \sum_{i=1}^{n} c_i\ \mathbf{X}_i\ e^{\lambda_i x}. \qquad \ldots(4a)$$

Thus $\mathbf{u}(x)$ is a solution of the differential equation (3b) and

$$\mathbf{u}(x_0) = \sum_{i=1}^{n} c_i e^{\lambda_i x_0} \mathbf{X}_i = \sum_{i=1}^{n} b_i \mathbf{X}_i = \mathbf{c}.$$

Now, let

$$\mathbf{w}(x) = \sum_{i=1}^{n} a_i(x) \mathbf{X}_i e^{\lambda_i x} \qquad \ldots(4b)$$

be a solution of the differential equation (1), where $a_1(x)$, $a_2(x)$, ..., $a_n(x)$ are scalar functions of x such that $a_i(x_0) = 0$.

Differentiating (4b) with respect to x, we get

$$\mathbf{w}'(x) = \sum_{i=1}^{n} a_i'(x) \mathbf{X}_i e^{\lambda_i x} + \sum_{i=1}^{n} a_i(x) \lambda_i \mathbf{X}_i e^{\lambda_i x}. \qquad \ldots(5)$$

Substituting (4b) and (5) in (1), we have

$$\sum_{i=1}^{x} a_i'(x) \mathbf{X}_i e^{\lambda_i x} + \sum_{i=1}^{n} a_i(x) \lambda_i \mathbf{X}_i e^{\lambda_i x} = \sum_{i=1}^{n} a_i(x) \mathbf{A} \mathbf{X}_i e^{\lambda_i x} + \mathbf{f}(x)$$

or

$$\sum_{i=1}^{n} a_i'(x) \mathbf{X} e^{\lambda_i x} = \sum_{i=1}^{n} a_i(x) [\mathbf{A}\mathbf{X}_i - \lambda_i \mathbf{X}_i] e^{\lambda_i x} + \mathbf{f}(x) = \mathbf{f}(x). \qquad \ldots(6)$$

Multiplying (6) by $\mathbf{Y}_j e^{-\lambda_j x}$ [where $\mathbf{Y}_1$, $\mathbf{Y}_2$, .., $\mathbf{Y}_n$ are left eigenvectors of $\mathbf{A}$ corresponding to the eigenvalues λ_1, λ_2, ..., λ_n] we get

$$\sum_{i=1}^{n} a_i'(x) \mathbf{Y}_j \mathbf{X}_i e^{(\lambda_i - \lambda_j)x} = \mathbf{Y}_j \mathbf{f}(x) e^{-\lambda_j x}$$

or

$$a_j'(x) \mathbf{Y}_j \mathbf{X}_j = \mathbf{Y}_j \mathbf{f}(x) e^{-\lambda_j x}, \; [\because \mathbf{Y}_j \mathbf{X}_i = 0 \text{ for } i \neq j]$$

or

$$a_j'(x) = \frac{1}{\mathbf{Y}_j \mathbf{X}_i} \mathbf{Y}_j \mathbf{f}(x) e^{-\lambda_j x}$$

or

$$a_j(x) = \int_{x_0}^{x} (\mathbf{Y}_j \mathbf{X}_j)^{-1} \ \mathbf{Y}_j \mathbf{f}(s) e^{\lambda_j s} ds \qquad \ldots(7)$$

$$[\therefore a_j(x_0) = 0 \text{ for } j = 1, 2. \ldots, n].$$

Now take

$$\mathbf{v}(x) = \mathbf{u}(x) + \mathbf{w}(x). \qquad \ldots(7a)$$

Differentiating we get

$$\begin{aligned} \mathbf{v}'(x) &= \mathbf{u}'(x) + \mathbf{w}'(x) \\ &= \mathbf{A}\,\mathbf{u}(x) + \mathbf{A}\,\mathbf{w}(x) + \mathbf{f}(x) \\ &= \mathbf{A}\,[\mathbf{u}(x) + \mathbf{w}(x)] + \mathbf{f}(x) \\ &= \mathbf{A}\,\mathbf{v}(x) + \mathbf{f}(x) \end{aligned}$$

and

$$\mathbf{v}(x_0) = \mathbf{u}(x_0) + \mathbf{w}(x_0) = \mathbf{c}.$$

Hence $\mathbf{v}(x) = \mathbf{u}(x) + \mathbf{w}(x)$ is the unique solution of the differential equation (1) satisfying the condition (2).

APPLICATIONS

We now proceed to apply the foregoing theory in the problems of thermoelastic and magnetothermoelastic interactions in an infinite medium.

I. *Thermoelastic interactions in an infinite solid with instantaneous heat source*

The coupled one dimensional equations of heat conduction and motion are given by

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t} \right) T = b_1 \frac{\partial^2 u}{\partial x \, \partial t} - d_1 Q(x, t) \qquad \ldots(8)$$

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right) u = a_1 \frac{\partial T}{\partial x} - \frac{k_1}{c_1 (\lambda + 2\mu)} F(x, t). \qquad \ldots(9)$$

The equations are written in the non-dimensional form, where the position x, the displacement u, the normal stress τ_{xx}, the time t and the temperature T are related to their dimensional counterparts x_1, u_1, $(\tau_1)_{xx}$, t_1 and T_1 by the relations $x = c_1 x_1/k_1$, $u = c_1 u_1/k_1$, $\tau_{xx} = (\lambda + 2\mu)^{-1} (\tau_1)_{xx}$, $t = c_1^2 t_1/k_1$ and $T = T_1/T_0$ respectively.

For the solution of eqns. (8) and (9), we shall use Laplace transforms defined by

$$\bar{u} = \int_0^\infty u \exp(-pt) \ dt, \ \bar{T} = \int_0^\infty \bar{T} \exp(-pt) \ dt.$$

Taking Laplace transforms of equations (8) and (9) and writing the resulting equations in the matrix form we obtain

$$\frac{d}{dx}\begin{bmatrix} \bar{T}(x,p) \\ \bar{u}(x,p) \\ \bar{T}'(x,p) \\ \bar{u}'(x,p) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ p & 0 & 0 & b_1 p \\ 0 & p^2 & a_1 & 0 \end{bmatrix}\begin{bmatrix} \bar{T}(x,p) \\ \bar{u}(x,p) \\ \bar{T}'(x,p) \\ \bar{u}'(x,p) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -d_1\bar{Q}\left(\frac{k_1}{c_1}x,\ \frac{k_1}{c_1^2}p\right) \\ -\frac{k_1}{c_1(\lambda+2\mu)}\bar{F}\left(\frac{k_1}{c_1}x,\ \frac{k_1}{c_1^2}p\right) \end{bmatrix} \qquad \ldots(10)$$

Where we have used the conditions that u, $\frac{\partial u}{\partial t}$ and T are initially zero. The prime indicates differentiation with respect to x.

Equation (10) can be written in contracted form as

$$\frac{d\mathbf{X}}{dx} = \mathbf{A}\ \mathbf{v} + \mathbf{f}(x) \qquad \ldots(11)$$

where $\mathbf{v}(x, p)$ denotes the state vector in the transformed domain whose components consists of transformed temperature, displacement as well as their gradients.

We now consider an infinite medium which is unstressed and unstrained initially but has a uniform temperature distribution T_0. It is then subjected to heat sources distributed over a plane area. The problem is to determine the subsequent distribution of temperature, stress and deformations as well as the interaction between the temperature and deformation fields.

Let $x_1 = \frac{k_1 x}{c_1} = 0$ represent the plane area over which the heat sources are situated and the solid medium occupies the whole space $-\alpha < x < +\infty$. We assume that the heat source is instantaneous and acts on the plane $r = 0$. We may represent it as

$$Q\left(\frac{k_1}{c_1}x,\ \frac{k_1}{c_1^2}t\right) = q\,\delta\left(\frac{k_1 x}{c_1}\right)\delta\left(\frac{k_1}{c_1^2}t\right)$$

So that

$$\bar{Q}\left(\frac{k_1}{c_1}x, \frac{k_1}{c_1^2}p\right) = q\,\frac{c_1^2}{k_1}\,\delta\left(\frac{k_1 x}{c_1}\right) \quad \text{...(12)}$$

where q is the constant strength of the source and $\delta\left(\frac{k_1}{c_1}x\right)$ is the Dirac delta function. The characteristic equation corresponding to the matrix $\mathbf{A}$ in equation (11) is given by

$$R^4 - p\,(p + 1 + \epsilon)\,R^2 + p^3 = 0 \quad \text{...(13)}$$

where $\epsilon = a_1 b_1$ is the coupling parameter. Let the roots of the equation (13) be $\pm R_1$, $\pm R_2$ which are also the eigenvalues of the matrix $\mathbf{A}$. A set of right and left eigenvectors $\mathbf{X}_i$, $\mathbf{Y}_i$ $(i = 1, 2, 3, 4)$ corresponding to the eigenvalues R_1, $-R_1$, R_2, $-R_2$ of $\mathbf{A}$ can be calulated as

$$X_1 = \begin{bmatrix} p^2 - R_1^2 \\ -a_1 R_1 \\ R_1(p^2 - R_1^2) \\ -R_1^2 a_1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} p^2 - R_1^2 \\ a_1 R_1 \\ -R_1(p^2 - R_1^2) \\ -R_1^2 a_1 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} p^2 - R_2^2 \\ -a_1 R_2 \\ R_2(p^2 - R_1^2) \\ -R_2^2 a_1 \end{bmatrix}, \quad X_4 = \begin{bmatrix} p^2 - R_2^2 \\ a_2 R_2 \\ -R_2(p^2 - R_2^2) \\ -R_2^2 a_1 \end{bmatrix} \quad \text{...(14a)}$$

$$\left.\begin{aligned} Y_1 &= \left[R_1^2 - p^2,\ p^2 b_1 R_1,\ \frac{R_1}{p}\left(R_1^2 - p^2\right),\ b_1 R_1^2\right], \\ Y_1 &= \left[R_1^2 - p^2,\ -p^2 b_1 R_1,\ -\frac{R_1}{p}\left(R_1^2 - p^2\right),\ b_1 R_2^2\right], \\ Y_3 &= \left[R_2^2 - p^2,\ p^2 b_1 R_2,\ \frac{R_2}{p}\left(R_2^2 - p^2\right),\ b_1 R_1^2\right], \\ Y_4 &= \left[R_2^2 - p^2,\ -p^2 b_1 R_2,\ -\frac{R_2}{p}\left(R_2^2 - p^2\right),\ b_1 R_2^2\right] \end{aligned}\right\} \quad \text{...(14b)}$$

For simplicity we assume that the body force $F(x, t) = 0$, and assuming other physical conditions of the problem as in Das *et al.*[11] and Paria[12], we write from (4a), (4b), (7), (7a), (12), (14a) and (14b)

$$\mathbf{v}(x) = a_2(x)\,\mathbf{X}_2\,e^{-R_1 x} + a_4(x)\,\mathbf{X}_4\,e^{-R_2 x} \quad \text{...(15)}$$

where

$$a_2(x) = \frac{1}{Y_2 X_2} \int_{x_0=-\infty}^{x} Y_2 \, \mathbf{f}(a) \, e^{-R_1 s} \, ds,$$

$$= \frac{1}{Y_2 X_2} \frac{q c_1^3 d_1 R_1 (R_1^2 - p^2)}{p k_1^2} .$$

$$a_4(x) = \frac{1}{Y_4 X_4} \int_{x_0=-\infty}^{x} Y_4 \, \mathbf{f}(s) \, e^{-R_2 s} \, ds$$

$$= \frac{1}{Y_4 X_4} \frac{q c_1^3 d_1 R_2 (R_2^2 - p^2)}{p k_1^2} .$$

Thus the displacement and the temperature fields can be written from (15) as

$$\bar{u}(x, p) = \frac{q c_1^3 d_1 R_1^2 (R_1^2 - p^2) a_1}{p k_1^2 V_2} e^{-R_1 x} + \frac{q c_1^2 d_1 R_2^2 (R_2^2 - p^2 a_1}{p k_1^2 V_4} \times e^{-R_2 x}, \; x > 0$$

$$\bar{T}(x, p) = - \frac{q c_1^2 d_1 R_1 (R_1^2 - p^2)^2}{p k_1^2 V_2} e^{-R_1 x} - \frac{q c_1^3 d_1 R_2 (R_2^2 - p^2)^2}{p k_1^2 V_4} e^{-R_2 x}, \; x > 0$$

where

$$V_2 = -\left(R_1^2 - p^2 \right)^2 - a_1 R_1^2 b_1 \left(p^2 + R_1^2 \right) - \frac{R_1^2 (R_1^2 - p^2)^2}{p}$$

$$V_4 = -\left(R_2^2 - p^2 \right)^2 - a_1 R_2^2 b_2 \left(p^2 + R_2^2 \right) - \frac{R_2^2 (R_2^2 + p^2)^2}{d} .$$

Using the characteristic equation (13), one can write the expressions for $\bar{u}(x, p)$ and $\bar{T}(x, p)$ in a simplified form

$$\bar{u}(x, p) = \frac{q a_1 c_1^3 d_1}{2 k_1^2 (R_1^2 - R_2^2)} [e^{-R_2 x} - e^{-R_1 x}], \; x > 0. \qquad \text{...(16)}$$

$$\bar{T}(x,p) = \frac{q c_1^3 d_1}{2 k_1^2 (R_1^2 - R_2^2) R_1 R_2} \left[R_2 \left(R_1^2 - p^2 \right) e^{-R_1 x} - R_1 \left(R_2^3 - p^2 \right) \times e^{-R_2 x} \right], \; x > 0. \qquad \text{...(17)}$$

Equations (16) and (17) are in complete agreement with Das *et al.*[11] and these equations. when written in the dimensional form, are also in complete agreement with Paria[12] in different notations.

II Magnetothermoelastic interactions in an infinite solid with instantaneous heat source

In the absence of displacement current and charge density, Maxwell's equations reduce to [Vide Paria[13]]

$$\text{rot } \mathbf{H} = \frac{4\pi}{c}\,\mathbf{j}, \ \text{rot } \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{ct_1} \qquad \text{...(18)}$$

$$\text{div } \mathbf{B} = 0, \ \mathbf{B} = \mu_e \mathbf{H},$$

while Ohm's law is given by

$$\mathbf{j} = \eta\left[\mathbf{E} + \frac{1}{c}\frac{\partial u_1}{\partial t_1} \times \mathbf{B}\right], \qquad \text{...(19)}$$

where we have neglected the effect of temperature on current.

The equation of heat conduction, neglecting the effect of current on temperature, is given by

$$k\nabla^2 T_1 + Q = pC_v \frac{\partial T_1}{\partial t_1} + T_0\beta \frac{\partial^2 u_1}{\partial x_1\,\partial t_1}. \qquad \text{...(20)}$$

Taking into account the Lorentz body force, the equation of motion becomes,

$$\rho\frac{\partial^2 (u_1)_i}{\partial t_1^2} = \frac{\partial (\tau_1)_{ik}}{\partial (x_1)_k} + \frac{1}{c}(\mathbf{j} \times \mathbf{B})_i + F. \qquad \text{...(21)}$$

The stress-strain relations are given by

$$(\tau_1)_{ij} = 2\mu\,(e_1)_{ij} - (\lambda e_1 - \beta T_1)\,\delta_{ij}$$

where

$$(e_1)_{ij} = \tfrac{1}{2}\{(u_1)_{ij} - (u_1)_{j,i}\}$$

$$e_1 = \text{div } (\mathbf{u}_1).$$

[subscript "1" indicates the quantities involved are dimensional].

Assuming for simplicity $\mu_e \approx 1$ (as is practically justified) the coupled one dimensional equations of magnetothermoelasticity in the non-dimensional form were given by Das and Bhattacharya[6], Paria[13] as

$$\left(\frac{\partial^2}{\partial x^2} - d\frac{\partial}{\partial t}\right) h_y = 0 \qquad \text{...(22)}$$

$$\left(\frac{\partial^2}{\partial x^2} - d\frac{\partial}{\partial t}\right)\left(\frac{h_z}{H_3}\right) = Q\frac{\partial^2 u}{\partial x\,\partial t} \qquad \text{...(22)}$$

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}\right) T = b_1\frac{\partial^2 u}{\partial x\,\partial t} - d_1\, q\left(\frac{k_1}{c_1}x, \frac{k_1}{c_1^2}t\right) \qquad \text{...(24)}$$

$$\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial t^2}\right)u=a_1\frac{\partial T}{\partial x}+b\frac{\partial}{\partial x}\left(\frac{h_z}{H_3}\right)-\frac{k_1}{c_1(\lambda+2\mu)}\times F\left(\frac{k_1}{c_1}x,\frac{k_1}{c_1^2}t\right). \quad ...(25)$$

Equation (22) gives $h_y=0$ since h_y is zero initially. Now for the solutions of eqns. (23)-(25) for the variables u, T and h_z we proceed as follows.

We consider an infinite elastic solid without any initial stress or strain but having an uniform temperature distribution T_0 throughout. The body is embedded in a uniform magnetic field of intensity H_3. The problem is to determine the distribution of stress, strain, temperature electric and magnetic field in the body, if it is subjected to instantaneous heat sources distributed on a plane extending in all directions. Without any loss of generality, the plane containing the heat sources can be taken parallel to the initial magnetic field of intensity H_3, and y_1-axis in the plane of the heat sources. The axis of x_1 is then perpendicular to this plane such that x_1, y_1, z_1 axis form a right handed system of cartesian co-ordinates. We take the heat source in the same form as in (12) and neglect the body forces. For simplicity, we assume that the medium is of infinite conductivity.

Letting $\eta\to\infty$ i. e. $\nu_H\to 0$ or $d\to\infty$, appropriate equations from (23)-(25) reduce, in Laplace transform domain, to

$$h_z=-H_3\frac{d\bar{u}}{dx} \quad ...(26)$$

$$\left(\frac{d^2}{dx^2}-p\right)\bar{T}=b_1p\frac{d\bar{u}}{dx}-d_1\bar{Q}\left(\frac{k_1}{c_1}x,\frac{k_1}{c_1^2}p\right) \quad ...(27)$$

$$\left(\frac{d^2}{dx^2}-p^2\right)\bar{u}=a_1\frac{d\bar{T}}{dx_1}+\frac{b}{H_3}\frac{d\bar{h}_z}{dx}. \quad ...(28)$$

Using $\bar{h}_z$ from (26) in (28), the resulting eqn. and the equation (27) can be written in the matrix form as

$$\frac{d}{dx}\begin{bmatrix}\bar{T}(x,p)\\ \bar{u}(x,p)\\ \bar{T}'(x,p)\\ \bar{u}'(x,p)\end{bmatrix}=\begin{bmatrix}0&0&1&0\\0&0&0&1\\p&0&0&b_1p\\0&\frac{p^2}{1+b}&\frac{a_1}{1+b}&0\end{bmatrix}\begin{bmatrix}\bar{T}(x,p)\\ \bar{u}(x,p)\\ \bar{T}(x,p)\\ \bar{u}'(x,p)\end{bmatrix}+\begin{bmatrix}0\\0\\-d_1\bar{Q}\left(\frac{k_1}{c_1}x,\frac{k_1}{c_1^2}p\right)\\0\end{bmatrix} \quad ...(29)$$

which can be written in the contracted form as

$$\frac{d\mathbf{v}}{dx} = \mathbf{A}\,\mathbf{v} + \mathbf{f}(x). \qquad \ldots(30)$$

The characteristic equation corresponding to the matrix **A** in (30) is given by

$$R^4 - \frac{p\,(p+1+\epsilon+b)}{1+b}\,R^2 + \frac{p^3}{1+b} = 0. \qquad \ldots(31)$$

Let $\pm R_1$, $\pm R_2$ be roots of the equation (31), then $R_1, -R_1, R_2$ and $-R_2$ are the eigenvalues of **A**.

A set of right and left eigenvectors $\mathbf{X}_i$, $\mathbf{Y}_i$ ($i = 1, 2, 3, 4$) corresponding to the eigenvalues $R_1, -R_1, R_2, -R_2$ of **A** can be calculated as

$$X_1 = \begin{bmatrix} -R_1^2 + \dfrac{p^2}{1+b} \\[2ex] \dfrac{-a_1 R_1}{1+b} \\[2ex] R_1\left(-R_1^2 + \dfrac{p^2}{1+b}\right) \\[2ex] \dfrac{-a_1 R_1^2}{1+b} \end{bmatrix} \quad X_2 = \begin{bmatrix} -R_1^2 + \dfrac{p^2}{1+b} \\[2ex] \dfrac{a_1 R_1}{1+b} \\[2ex] -R_1\left(-R_1^2 + \dfrac{p^2}{1+b}\right) \\[2ex] \dfrac{-a_1 R_1^2}{1+b} \end{bmatrix}$$

$$X_3 = \begin{bmatrix} -R_2^2 + \dfrac{p^2}{1+b} \\[2ex] \dfrac{-a_1 R_2}{1+b} \\[2ex] R_2\left(-R_2^2 + \dfrac{p^2}{1+b}\right) \\[2ex] \dfrac{-a_1 R_2^2}{1+b} \end{bmatrix} \quad X_4 = \begin{bmatrix} -R_2^2 + \dfrac{p^2}{1+b} \\[2ex] \dfrac{a_1 R_2}{1+b} \\[2ex] -R_2\left(-R_2^2 + \dfrac{p^2}{1+b}\right) \\[2ex] \dfrac{-a_1 R_2^2}{1+b} \end{bmatrix} \qquad \ldots(32a)$$

$$\left.\begin{aligned} Y_1 &= \left[R_1^2 - \frac{p^2}{1+b},\ \frac{b_1 p^2 R_1}{1+b},\ \frac{R_1}{p}\left(R_1^2 - \frac{p^2}{1+b}\right),\ b_1 R_1^2\right], \\ Y_2 &= \left[R_1^2 - \frac{p^2}{1+b},\ \frac{-b_1 p^2 R_1}{1+b},\ -\frac{R_1}{p}\left(R_1^2 - \frac{p^2}{1+b}\right),\ b_1 R_1^2\right] \\ Y_3 &= \left[R_2^2 - \frac{p^2}{1+b},\ \frac{b_1 p^2 R_2}{1+b},\ \frac{R_2}{p}\left(R_2^2 - \frac{p^2}{1+b}\right),\ b_1 R_2^2\right] \\ Y_4 &= \left[R_2^2 - \frac{p^2}{1+b},\ \frac{-b_1 p^2 R_2}{1+b},\ \frac{-R_2}{p}\left(R_2^2 - \frac{p^2}{1+b}\right)\ b_2 R_2^2\right] \end{aligned}\right\}. \qquad \ldots(32b)$$

Assuming the physical conditions to be the same as in Paria[13] and Das *et al.*[11] of the problem, we write from (4a), (4b), (7), (7a), (12), (32a) and (32b);

$$\mathbf{v}(x) = a_2(x)\,\mathbf{X}_2\, e^{-R_1 x} + a_4(x)\,\mathbf{X}_4\, e^{-R_2 x} \qquad \ldots(33)$$

where

$$a_2(x) = \frac{1}{Y_2 X_2}\int_{x_0=-\infty}^{x} Y_2\,\mathbf{f}(s)\, e^{-R_1 s}\, ds,\ x > 0$$

$$= \frac{1}{Y_2 X_2}\,\frac{q\, c_1^3\, d_1\, R_1\left(R_1^2 - \dfrac{p^2}{1+b}\right)}{p k_1^2}$$

$$a_4(x) = \frac{1}{Y_4 X_4}\int_{-\infty}^{x} Y_4\, f(s)\, e^{-R_2 s}\, ds$$

$$= \frac{1}{Y_4 X_4}\,\frac{q c_1^2\, d_1\, R_2\left(R_2^2 - \dfrac{p^2}{1+b}\right)}{p k_1^2},\ x > 0$$

and

$$Y_2 X_2 = -\left(R_1^2 - \frac{p^2}{1+b}\right)^2\left(1 + \frac{R_1^2}{p}\right) - \frac{a_1 b_1 R_1^2}{1+b}\left(\frac{p^2}{1+b} + R_1^2\right),$$

$$Y_4 X_4 = -\left(R_2^2 - \frac{p^2}{1+b}\right)^2\left(1 + \frac{R_2^2}{p}\right) - \frac{a_1 b_1 R_2^2}{1+b}\left(\frac{p^2}{1+b} + R_2^2\right).$$

As in the previous application we now use the characteristic equation (31) and write the displacement and temperature fields from (33) in a simplified form as follows :

$$\bar{u}(x, p) = \frac{q c_1^3\, d_1\, a_1}{2k_1^2\,(R_1^2 - R_2^2)(1+b)}\left(e^{-R_2 x} - e^{-R_1 x}\right),\ x > 0 \qquad \ldots(34)$$

$$\bar{T}(x, p) = \frac{q c_1^3\, d_1}{2k_1^2\, R_1\, R_2\,(R_1^2 - R_2^2)}\Bigg[R_2\left(R_1^2 - \frac{p^2}{1+b}\right) e^{-R_1 x}$$

$$- R_1\left(R_2^2 - \frac{p^2}{1+b}\right) e^{-R_2 x}\Bigg],\ x > 0. \qquad \ldots(35)$$

The perturbation of the magnetic field can now be determined from (26) as

$$\bar{h}_z(x, p) = -\frac{H_3\, q\, c_1^3\, d_1\, a_1}{2(1+b)\, k_1^2\,(R_1^2 - R_2^2)}\left[R_1 e^{-R_1 x} - R_2 e^{-R_2 x}\right],\ x > 0 \qquad \ldots(36)$$

It may be noticed that in the absence of the magnetic field i.e. when $b = 0$, eqns. (34) and (35) are identical with (16) and (17).

Paria[12] solved the same problem by the method of Laplace-Fourier transforms and found out the expression for $\bar{h}_z(x, p)$ in eqn. (10 36a) in the Laplace transform domain, but this expression is not in agreement with our expression (36) due to a mistake in his calculation in eqn. (10.15d).

4. Numerical Inversion and Analysis of the Results

The numerical inversion in the space time domain of the displacement and temperature fields of the problem in application I i. e. the inversions of eqns. (16) and (17) have been made by Paria[12] for small time. Paria[13] and Das *et al*[11]. presented the distribution of stress and induced magnetic field for small time of the problem in application II.

We now propose to invert numerically the temperature fields for both the problems in applications I and II by Bellman method for any time interval. Although we have made a particular choice of the time interval and compared the magnetic effect upon the temperature distribution in the two problems, the comparison can be made for any time interval by the shifting theorem of Laplace transform, vide Bellman[14].

In order to compare the interactions of the magnetic field with the thermoelastic field is the deformation and temperature; we now propose the medium to be copper, and thus $\epsilon = 0.003722$ in MKS units, and take $x = 1$, and invert eqns. (16), (17), (34) and (35) for a specific time range and for two different values of the intensity of the magnetic field as shown in the table. Finally, the inversion for the perturbation of the magnetic field, h_z in eqn. (36), has been made for the same two values of the intensities for comparison. The computations have been made in Burroughs 6700 of the Regional Computer Centre, Jadavpur University, Calcutta.

It is clear from Table I that the displacement, temperature and the perturbation of the original magnetic field have the small oscillatory character about the time axis and they all decrease with the intensity of the magnetic field, as expected. It is also clear that the effect of the magnetic field is not much pronounced on the temperature field in comparison to the displacement field and the perturbation of the original magnetic field.

Concluding Remarks

It is clear from foregoing analysis that the present approach can be very easily applied to a broad class of boundary value problems is mechanics and other fields where solution of simultaneous, coupled, differential equations is required. The state space methodology of Bahar and Hetnarski[1] for homogeneous equations and of Das *et al.*[11] for non-homogeneous equations converts the boundary value problem to an initial value problem before its solution. Thus the final solution requires very extensive use of algebra for the determination of these initial values through boundary condition by using the matrix equations of the influnce functions. In the present approach, the original structure of the problem is retained[8].

TABLE I

In the table the numerical values, as obtained from the computer, are rounded off to six significant figures

Eqn. no.	t	0.0257750	0.138382	0.352509	0.693147	1 21376	2.04612	5.67119
16	$\frac{2k_1^2 \; u(1,t)}{qa_1 \, c_1^3 \, d_1}$ $b = 0$	−0.182549E+05	0.256421E+05	−0.293849E+05	0.319915E+05	−0.354164E+05	0.423166E+05	−0.626672E+5
34	$\frac{2k_1^2 \; u(1,t)}{qc_1^3 \, d_1 a_1}$ $b = 0.01$	−0.174558E+05	0.245012E+05	−0.280516E+05	0.305143E+05	−0.337593E+05	0.403193E+05	−0.596955E+05
34	$\frac{2k_1^2 \; u(1,t)}{qc_1^3 \, d_1 \, a_1}$ $b = 0.1$	−0.119751E+05	0.166968E+05	−0.189622E+05	0.204774E+05	0.225301E+05	0.268085E+05	−0.396133E+05
17	$\frac{2k_1^2 \; T(1,t)}{qc_1^3 \, d_1}$ $b = 0$	−0.423791E+03	0.4117128+03	−0.348171E+03	0.315162E+03	−0.315661E+03	0.355571E+03	−0.513241E+03
35	$\frac{2k_1^2 \; T(1,t)}{qc_1^3 \, d_1}$ $b = .01$	−0 420903E+03	0.407584E+03	−0.343340F+03	0.309802E+03	−0.309644E+03	0.348315E+03	−0.502442E+03
35	$\frac{2k_1^2 \; T(1,t)}{qc_1^3 \, d_1}$ $b = 0.1$	−0.400634E+03	0 378679E+03	−0.309623E+03	0.272525E+03	−0.267902E+03	0.298063E+03	−0.427729E+03

(*Table continued on p.* 711)

Eqn. no.	t	0.0257750	0.138382	0.352509	0.693147	1.21376	2.04612	3.67119
36	$\dfrac{2k_1^2\, h_z(1,t)}{H_3\, qc_1^3\, d_1 a_1}$ $b = 0.01$	$0.160832E+06$	$-0.229348E+06$	$0.2676\ E+06$	$-0.296158E+06$	$0.331901E+06$	$0.399815E+06$	$0.594666E+06$
36	$\dfrac{2k_1^2\, h_z(1,t)}{H_3 qc_1^3\, d_1 a_1}$ $b = 0.1$	$0.106986E+06$	$-0.151768E+06$	$0.179908E+06$	$-0.193395E+06$	$0.215633E+06$	$0.258852F+07$	$0.384274E+06$

We hope to communicate, in future, the theory and application for the case when the eigenvalues of the matrix **A** in (1) are repeated.

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MAGNETO-ELASTIC TRANSVERSE SURFACE WAVES IN SELF-REINFORCED ELASTIC SOLIDS

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When the direction of a uniform magnetic field is different from that of wave propagation, the surface wave of the SH type can also be propagated in a self reinforced, anisotropic elastic half space without dispersion, as in isotropic case. Reinforcement happens only to increase the decay rate of the waves.

1. INTRODUCTION

Idea, here, is to examine whether it is possible to propagate the uncoupled surface waves of SH type without showing dispersion[1] in a perfectly conducting anisotropic elastic half space, endowed with self reinforcement[2] in the presence of a uniform magnetic field or not. We find, as in elastic case[3], that when the plane of the magnetic field and the direction of wave propagation is normal to free surface, it is possible provided the electromagnetic radiation into the adjacent free space is not neglected. The role of reinforcement is merely to increase the decay rate of the waves.

2. BASIC EQUATIONS

(i) Stress-strain law governing self reinforced elastic medium[2] whose preferred direction is that of a unit vector **a** is given by

$$t_{ij} = \lambda\, e_{kk}\, \delta_{ij} + 2\mu_T\, e_{ij} + \alpha\,(a_k\, a_m\, e_{km}\, \delta_{ij} + e_{kk}\, a_i\, a_j)$$
$$+ 2\,(\mu_L - \mu_T)\,(a_i\, a_k\, e_{kj} + a_j\, a_k\, e_{ki}) + \beta'\, a_k\, a_m\, e_{km} a_i\, a_j. \quad \ldots(2.1)$$

In (2.1), λ, μ_T, μ_L,β' and α are all elastic constants with the dimensions of stress. t_{ij} is stress tensor e_{ij} is the strain tensor. a_i are the components of **a**, the unit vector denoting the preferred direction. All suffixes take values from 1 to 3 and repeated suffix means summation. We are taking the physical components of the respective tensors. δ_{ij} is the kronecker delta.

(ii) The electromagnetic field equations for free space are

$$\nabla^2 \mathbf{h} - \frac{1}{C^2}\frac{\partial^2 \mathbf{h}}{\partial t^2} = 0,\ \text{Curl}\ \mathbf{h} = \epsilon_0 \frac{\partial \mathbf{e}}{\partial t} \quad \ldots(2.2)$$

$$\nabla^2 \mathbf{e} - \frac{1}{C^2}\frac{\partial^2 \mathbf{e}}{\partial t^2} = 0,\ \text{Curl}\ \mathbf{e} = -\mu_0 \frac{\partial \mathbf{h}}{\partial t}\,. \quad \ldots(2.3)$$

In (2.2) to (2.3) **e** and **h** are perturbations in the electric and magnetic field vectors within the medium, given by

$$e = -\mu_e \left(\frac{\partial \mathbf{u}}{\partial t} \times \mathbf{H} \right) \mathbf{h} = \mathrm{Curl}\left(\mathbf{u} \times \mathbf{H} \right) \qquad \text{...(2.4)}$$

where **u** is the displacement vector, **H** is the primary magnetic field and μ_e is the magnetic permeability. In (2.2) and $(2.3)_{11}$ $C = (\epsilon_0 \mu_0)^{-1/2}$ denotes the velocity of light, ϵ_0 and μ_0 being the electric and magnetic parmeabilities of the free space.

(iii) The equation of motion is

$$t_{ij,j} + (J \times B)_i = \rho \frac{\partial^2 u_i}{\partial t^2}. \qquad \text{...(2.5)}$$

In (2.5), u_i are the components of displaeement **u**. **J** and **B** are current density and magnetic induction vectors respectively. We have

$$\mathbf{B} = \mu_e \mathbf{H}, \mathbf{J} = \sigma \left[\mathbf{E} + \frac{\partial \mathbf{u}}{\partial t} \times \mathbf{B} \right]. \qquad \text{...(2.6)}$$

(iv) Boundary conditions to be satisfied at the free boundaries of separation are

$$\left[\left[\mathbf{e} + \frac{\partial \mathbf{u}}{\partial t} \times \mathbf{B} \right]\right]_t = 0, \left[\left[\mathbf{B} \right]\right]_n = 0 \qquad \text{...(2.7)}$$

$$(t_{ij} + M_{ij}) \nu_j = \overline{M}_{ij} \nu_j, \qquad \text{...(2.8)}$$

where M_{ij} and $\overline{M}_{ij}$ are the Maxwell stress tensors for the material and for the free space respectively, ν_j is the unit normal vector at a point of the boundary. A double bracket with a suffix t and n in (2.7) means difference between the tangential and the normal components.

3. Surface Waves of SH Type

We consider the propagation of harmonic waves in a self-reinforced anisotropic elastic medium occupying the semi-infinite space $Z \geqslant 0$ (the plane $Z = 0$ which separates the medium from the free space, is free of mechanical tractions) in the presence of a uniform magnetic field. We take the displacement and magnetic field components in the form

$$u = 0,\ v = A \exp\ (pz + ik(x - Vt)),\ \omega = 0 \qquad \text{...(3.1)}$$

$$H_x = H \cos \phi,\ H_y = 0,\ H_z = H \sin \phi, \qquad \text{...(3.2)}$$

ϕ being the angle at which the wave crosses the magnetic field. The components of reinforcement director are

$$\mathbf{a} = (a_1, 0, a_3). \qquad \text{...(3.3)}$$

For free space, the elastic and magnetic field perturbations are

$$e = (\bar{e}_1, 0, \bar{e}_3)\, e^{qz+ik(x-Vt)} \qquad \text{...(3.4)}$$

$$h = (0, \bar{h}_2, 0)\, e^{qz + ik(x - Vt)} \quad \ldots(3.5)$$

the constants $\bar{e}$ and $\bar{h}$ being connected by

$$\bar{e}_1 = -\frac{iq}{\epsilon_0 kV}\bar{h}_2, \quad \bar{e}_3 = -\frac{1}{\epsilon_0 V}\bar{h}_2. \quad \ldots(3.6)$$

In (3.4) to (3.6), q being given by

$$q^2 = k^2\left(1 - \frac{V^2}{C^2}\right) \quad \ldots(3.7)$$

is to be taken with proper sign so that electromagnetic radiation into free space does not vanish for $Z \to -\infty$.

4. Solution

The equations determining v and h_2 are

$$\begin{bmatrix} p\, H \sin\phi + ik\, H\cos\phi & -1 \\ p^2\beta^2 + k^2(V^2 - \beta^2) & k/\rho\,(p\, H\sin\phi + ik\, H\cos\phi) \end{bmatrix}\begin{bmatrix} v \\ h_2 \end{bmatrix} = 0. \quad \ldots(4.1)$$

Equation in p becomes

$$\left[\beta^2 + V_A^2 \sin^2\phi + (\mu_L - \mu_T)\frac{1}{\rho}a_3^2\right]p^2 + i\,kp\left[V_A^2 \sin 2\phi + 2a_1 a_3 \frac{\mu_T - \mu_T}{\rho}\right] + k^2\left[V^2 - \beta^2 - V_A^2\cos^2\phi - a_1^2\left(\frac{\mu_L - \mu_T}{\rho}\right)\right] = 0 \quad \ldots(4.2)$$

where $\beta = \sqrt{\mu_T/\rho}$ is the velocity of plane shear waves and $V_A = \sqrt{\mu_T/\rho}\; H$ is the Alfven velocity. Out of the two roots of (4.2), we choose one with a negative real part as for surface waves, displacement decays with increasing depth.

Equation (2.7) gives

$$\bar{e}_1 = ik\, V\, \mu_e\, H\sin\phi\, v,\; Z = 0. \quad \ldots(4.3)$$

The condition that the surface $Z = 0$ is free from mechanical tractions reduces to

$$p\left[\beta^2 + V_A^2\sin^2\phi + \frac{\mu_L - \mu_T}{\rho}a_3^2 + ik\left[\tfrac{1}{2}V_A^2\sin 2\phi + \frac{\mu_L - \mu_T}{\rho}a_1 a_3\right] + V_A^2\frac{V^2 k^2 \sin^2\phi}{qc^2}\right] = 0,\quad V_{A_0}^2 = \frac{\mu_e^2 H^2}{\mu_0\rho} \quad \ldots(4.4)$$

Elimination of p between (4.2) and (4.4) gives us

$$V^4\left[1 - \frac{V_{A_0}^4 \sin^4\phi}{C^2 I}\right] - V^2\left[V^2 + \frac{Q}{I}\right] + C^2\frac{Q}{I} = 0 \quad \ldots(4.5)$$

where

$$Q = \beta^2 \left(\beta^2 + V_A^2\right) + \{(\mu_L - \mu_T)/\rho\} \left\{\beta^2 + V_A^2 (a_1 \sin\phi - a_3 \cos\phi)^2\right\} \quad \ldots(4.6)$$

$$I = V_A^2 \sin^2\phi + \beta^2 + \{(\mu_L - \mu_T)/\rho\}\, a_3^2 \,. \quad \ldots(4.7)$$

To first approximation in $1/C$, the roots of (4.5) are

$$V_1^2 = C^2 \left(1 - \frac{V_{A_0}^4 \sin^4\phi}{C^2 I}\right), \; V_2^2 = \frac{Q}{I}\left(1 - Q\,\frac{V_{A_0}^3 \sin^4\phi}{C^4\, I^2}\right). \quad \ldots(4.8)$$

5. Discussion of the Results

(i) For V_1, q becomes imaginary and p has zero real part as is clear from equation (4.4). Hence this value is not suitable. The second value V_2 satisfies all the conditions. Thus purely transverse surfaces waves can be propagated, without any dispersion, in a conducting elastic but anisotropic half space, provided there exists an external magnetic field non alligned to the direction of wave propagation. Waves propagate with velocity V_2 given by $(4.8)_2$. Penetration depth is given by

$$\left\{d = \{qC^2\, V_A^2 \sin^2\phi + \beta^2 + (\mu_L - \mu_T)/\rho a_3^2\right\} / V_{A_0}^2\, V^2\, k^2 \sin^2\phi.$$

(ii) The reinforcement increases the strength of the transverse surface waves which decay out more quickly. Energy of the wave gets dissipated within a shorter longitudinal span.

For infinite thickness and conductivity of the plate, the velocity of the wave obtained in Verma[4] becomes the same as the one derived from the present solution on the assumption of infinite conductivity.

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ERRATA

Approximation of a function by the $F(a, q)$ transform of its Fourier series

by

M. S. Rangachari and S. A. Settu

Indian J. pure appl. Math., 19 (4): 369-383, *April* 1988

page	line	for	read
369	9	Kreisverfahran	Kreisverfahren
370	3	$q\,(q, k)$	$g\,(q, k)$
	10	(1,8)	(1.8)
	16	$\Upsilon\lambda$	$\sqrt{\lambda}$
	1 ↑	$\omega_f \geqslant \omega$	$\omega_f \leqslant \omega$
371	5	$\pi/m+\frac{1}{2}$	$\pi/(m+\frac{1}{2})$
	9 ↑	$\begin{matrix}(p+k)\\ k\end{matrix}$	$\binom{p+k}{k}$
	1 ↑	$S^p_\beta\,(x) = f\,(x)$	$S^p_\beta\,(x) - f\,(x)$
372	7	$3\gamma/2$	$\frac{3}{2}-\gamma$
	10	$\lvert r > \lvert q^\gamma$	$\lvert r \rvert > q^\gamma$
	3 ↑	$a_q{}^{2\gamma-1}$	$aq^{2\gamma-1}$
	1 ↑	$c^{-\lambda}$	$e^{-\lambda}$
3 73	9 ↑	$0\ \omega\left(\frac{1}{m}\right))$	$0\left(\omega\left(\frac{1}{m}\right)\right)$
	8 ↑	PROOG	PROOF
	3 ↑	$\sin\,(m+\frac{1}{2})\,t\,\rvert$	$\lvert \sin\,(m+\frac{1}{2})\,t\,\rvert$
	2 ↑	(t)	t
375	2 ↑	$v\,(m) - uu\,(m)$	$v\,(m) - u\,(m)$
	2 ↑	sin	$\sin\,(m+\frac{1}{2})\,t\,dt$
	1 ↑	$(m+\frac{1}{2})\,td\,t = I_1 + I_2+I_3+I_4+I_5$	$= I_1+I_2+I_3+I_4+I_5$
376	8	$\exp\,-mt^2/4a)$	$\exp\,(-mt^2/4a)$
377	7	$\sqrt{M}\omega\left(\frac{1}{m}\right.$	$\sqrt{m}\,\omega\left(\frac{1}{m}\right)$
	9	$\left(1+\frac{u\,(m)^{1/2}}{v\,(m)}\right.$	$\left(1+\frac{u\,(m)}{v\,(m)}\right)^{1/2}$
	6 ↑	$\rho\theta^{-i\theta} = 1-\beta e^{-it}$	$\rho e^{-i\theta} = 1-\beta e^{it}$
	3 ↑	$\left(\frac{1-\beta}{\rho}\right)^p \leqslant \text{exy}\,(-Apt^2)$	$\left(\frac{1-\beta}{\rho}\right)^p \leqslant \exp\,(-Apt^2)$

	1↑	$\left(\frac{1-\beta}{\rho} \right)^{p}$	$\left(\frac{1-\beta}{\rho}\right)^{p}$
	1↑	$\exp\left(\frac{1}{2}\, p\beta \left(\frac{t}{1-\beta} \right)^{2} \right)$	$\exp\left(-\frac{1}{2}p\beta\left(\frac{t}{1-\beta}\right)^{2}\right)$
378	2↑	S_3	S_4
	2↑	$\phi_x(t)$	$\lvert \phi_x(t) \rvert$
	2↑	$g(d, k)$	$g(q, k)$
379	3↑	$\omega \frac{1}{m})$	$\omega\left(\frac{1}{m}\right)$
380	8	$\int_0^\pi \frac{\lvert \phi_x(t)}{t}$	$\pi\int_0^\pi \frac{\lvert \phi_x(t) \rvert}{t}$
	9	$\exp(aq^{2\gamma-1}$	$\exp(-aq^{2\gamma-1})$
381	2	$\underset{m\leqslant k\leqslant +m\gamma}{+}$	$\underset{m\leqslant k\leqslant m+m\gamma}{\Sigma}$
	6	$\int_{1/m}^{1/m}$	$\int_{0}^{1/m}$
	3↑	$\binom{p-k}{p}$	$\binom{p+k}{p}$
382	1,2	$\left(\frac{1-\beta}{\rho}\right)^{p+1}$	$\left(\frac{1-\beta}{p}\right)^{p+1}$
	4↑	$\exp(-Ap+1)[b(p)\}^{2})$	$\exp(-A(p+1)[b(p)]^{2})$

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For Books

2. H. Rund, *The Differential Geometry of Finsler Spaces*, Springer-Verlag, Berlin, (1973) p. 283.

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